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A DISTINGUISHED CONTRIBUTOR TO THE MONTHLY

The name, Victor Thébault, is familiar to every reader of this MONTHLY. As a prolific and ingenious contributor to its pages, he has been the subject of many inquiries sent by readers to members of the editorial staff. This tripartite tribute to Monsieur Thébault has been written by three men who have observed his work through the years with great interest.—EDITOR.

VICTOR THÉBAULT—THE MAN

COL. W. E. BYRNE, Virginia Military Institute

In 1932, Victor Thébault became a member of the Mathematical Association of America. It was not long before his name appeared frequently among the contributors to the problem department. Since then many readers of this MONTHLY have wondered about the author of so many interesting problems and discussions.

Victor Michel Jean-Marie Thébault was born on March 6, 1882, at Ambrières-le-Grand (Mayenne), France. His father was a weaver. At the local primary school his teacher noted his native ability and took steps to obtain for him a scholarship at the teacher's college in Laval (Mayenne), where he remained as a student from 1898 to 1901. After graduation he became a school teacher at Pré-en-Pail (Mayenne) (1902–1905) until he was called to be a professor at the technical school of Ernée (Mayenne). In 1909, he received his certificate of capacity for a professorship (scientific) in teachers' colleges; this resulted from winning first place in competitive examination.

The modest salary of a professor was not sufficient to care for his family, now blessed with six children. Therefore Monsieur Thébault reluctantly gave up teaching to become a factory superintendent at Ernée (1910–23). This was followed by another change, to the position of Chief Insurance Inspector at Le Mans (Sarthe) (1924–40). In 1940 he retired to live at "Le Paradis" in Tennie (Sarthe).

That the farewell to teaching of mathematics did not mean any loss of interest in that field is attested by the honor conferred upon him by the French government. In 1932 he was made an Officier de l'Instruction Publique upon the recommendation of M. d'Ocagne, Member of the Institute, with the citation: "Personally I hold him in high esteem for his outstanding talent as a mathematician as shown by the numerous ingenious contributions to what is called elementary geometry, an unending source of problems whose solution requires a quite special gift of invention."

In 1935, he was made a Chevalier de l'Ordre de la Couronne of Belgium because of his activity in connection with the Scientific Society of Bruxelles, its *Annales*, and *Mathesis*.

Not only has Monsieur Thébault continued his publications (which include 15 communications to the Paris Académie des Sciences, hundreds of memoirs and articles concerning the modern geometry of the triangle and tetrahedron and the theory of numbers, and more than 1000 original problems), but also he

has contributed articles to the new French *Intermédiaire des Recherches Mathématiques*, and put his own library and services at the disposition of other mathematicians. Quite typical of the man is a notice appearing in the June, 1946, *Mathesis*: "M. V. Thébault, desirous of giving a new proof of his interest in *Mathesis*, offers to foreign subscribers the means of facilitating currency operations by accepting payment of their subscriptions at his postal checking account 339-03, Rennes, France." Moreover he established in 1943 the Prix Victor Thébault to be awarded every two years by the Paris Académie des Sciences to the author, preferably a teacher of the primary or secondary system, of an original study or of an interesting work on geometry or number theory.

Those who have had an occasion, as I have had, to correspond with Monsieur Thébault, have been greatly impressed by his charm and tact as well as by his keen interest in mathematics.

THÉBAULT—THE NUMBER THEORIST

E. P. STARKE, Rutgers University

M. Thébault's prolific output of theorems and problems about numbers is a source of constant admiration, little short of wonder, on the part of all those upon whom the higher arithmetic exerts its fascination. Since 1934 this MONTHLY and the *National Mathematics Magazine* have published upwards of a hundred thirty of these little gems. Few tools are needed to handle these problems beyond the simplest ideas of scales of notation, congruences, and occasionally the solution of a Pell equation, but there is also required a certain degree of analytical skill and considerable insight into numerical relations.

The character of a large majority of these problems may be judged from a few samples.

Determine the largest and smallest perfect squares which can be written with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, used once each in both cases. (E404. 1940, 48.)

In the scale of B there is a perfect square of sixteen digits of the form $abcdefghabcdefgh$. What is the smallest possible value of B? (E442. 1940, 656.)

Find three positive integers x, y, z, such that the sum of the ratios $y/z+z/x+x/y$ is an integer. (E682. 1945, 395.)

In what system of numbers with the base less than 10,000 are there the greatest number of squares of four digits of the form $aabb=(cc)^2$? (4184. 1945, 582.)

At first sight it might seem that so many problems of the same general nature could not all be interesting. On the other hand, however, the fascination increases and the little differences become as exciting to the student of the higher arithmetic as the next cross-word puzzle can be to the cross-word puzzle devotee.

No apology is needed for the expenditure of time and effort on these interesting little items. However, it seems appropriate to quote M. Thébault's own modest statement of his feelings on this point (*Mathésis* 1943, supplement).

"Some mathematicians exhibit a tendency, not altogether free from a certain disdain, to see in such problems only insignificant trifles. Trifles, if you please, but the solution of which often demands no less penetration of mind, ingenuity, and subtle artifice than many questions of allegedly deeper significance. Moreover the study of an elementary proposition sometimes necessitates an effort which is far from negligible, which constitutes an excellent intellectual exercise, and which leads to something worthwhile indeed.

"It is a fact that not all the great masters of the science have professed similar disdain for mathematical recreations. [Here M. Thébault gives quotations from Euler, Jacobi, and others.]

"This discipline by which we personally have been inspired has incited us to publish in a number of mathematical reviews a great many arithmetical curiosities in the form of notes or proposed problems which invited the reader to pause there with interest if not with delight. We have had the satisfaction of hearing it said that this modest contribution to the life of these journals, and to that extent to the mathematical growth of their readers, has been one of some service."

THÉBAULT—THE GEOMETER

N. A. Court, University of Oklahoma

When Victor Thébault appeared on the mathematical scene during the first decade of the present century, the geometry of the triangle had already left behind its "period of mushroom growth." It was a well established and extensive body of doctrine which commanded the loyalty of a considerable number of enthusiastic workers.

During the ensuing years the keen imagination of M. Thébault found in this domain a fertile field for the development of his powers. Practically every part of this discipline has attracted his attention, from the most modest to the most abstruse. He dealt with equilateral triangles as well as with conics and cubics associated with the triangle, with quadrilaterals and polygons, with the various remarkable points of the triangle, with the orthopole, the isopole, and so on.

An eloquent comment upon the quality and originality that characterized Thébault's work from the very start is the fact that J. L. Coolidge included in his renowned and much quoted *Treatise on the Circle and the Sphere* (1916) a proposition of Thébault (p. 113) that came to his notice after the corresponding chapter of the book had already been completed.

Some elements associated with the triangle have had a particular appeal for M. Thébault, and he returned to them time and again. Among them is Feuerbach's theorem. He considered Feuerbach's points in the very first article he contributed to the *Nouvelles Annales de Mathématiques* (series 4, vol. 10, 1910, pp. 271–281). Subsequently he devoted half a dozen papers to this topic, at various times, in which he exhibited a wealth of properties of these points. As late as 1935, he published a paper *Sur les points de Feuerbach* in the *Annales*

de la Société scientifique de Bruxelles, series A, vol. 55, pp. 39, in which he proves the following curious proposition: In a triangle (T) the four triangles having for vertices the four Feuerbach points of (T) are similar to the four triangles having for vertices the feet of the bisectors of (T).

As a side line, Thébault studied the "Shoemaker's knife" in several papers which appeared in various periodicals (*l'Enseignement Mathématique*, *Bulletin de la Société mathématique de France*, and so on).

The many and lasting contributions that M. Thébault made to the geometry of the triangle could not keep his inquiring mind from becoming attracted by the much less developed geometry of the tetrahedron which held out greater promise, even if it offered greater difficulty. His first paper dealing with the tetrahedron appeared in *Mathesis* in 1922. Of the many articles he has published since on this subject, articles in which he obtained a variety of important properties, the most important are those in which he associated with the tetrahedron spheres analogous to well-known circles associated with the triangle.

In 1932, he introduced the Longchamps sphere of the tetrahedron. In 1935 he added the Lucas spheres. Then came in rapid succession the Adams sphere, the Tucker sphere, the second Lemoine sphere. The adding of these spheres to the elements connected with the tetrahedron is undoubtedly a milestone in the development of the theory of the tetrahedron and promises to stimulate further advances. It is remarkable that this fruitful activity of M. Thébault took place during the sinister occupation of France by the Nazi hordes.

An outstanding feature of the prolific activities of Thébault is his interest in problems. There is hardly a periodical published anywhere with a problem department to which he has not contributed. The total number of questions proposed by him is counted by the hundreds and his record in this respect has few rivals. During the last decade or so American readers enjoyed the benefit of his contributions in this line both in the MONTHLY and in the *National Mathematics Magazine*. The originality and the variety of these questions have been the object of general admiration.

The quality and quantity of Thébault's geometrical contributions have earned for him a distinguished and enduring place among the most notable continuators of the work of Brocard, Lemoine, and Neuberg, the founders of the geometry of the triangle and the tetrahedron.

CONCERNING THE EULER LINE OF A TRIANGLE*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. **Notation.** Consider a triangle ABC inscribed in a circle (O), of center O and radius R. Let G be the centroid, H the orthocenter, N the center of the nine-point circle (N); A_1, B_1, C_1, the midpoints of the sides $BC=a$, $CA=b$, $AB=c$; A', B', C', the feet of the altitudes on these sides; S the area of the triangle. Unless otherwise stated we limit our discussion to the case of a triangle ABC with acute angles, not all three equal, leaving the examination of the other cases to the reader.

2. **A homothetic transform of the orthic triangle.** The centers O_a, O_b, O_c of the circumcircles of the triangles BOC, COA, AOB, of radii R_a, R_b, R_c, coincide with the intersections of the perpendicular bisectors of BC, CA, AB and those of OA, OB, OC, and

$$R_a = R/2\cos A, \qquad R_b = R/2\cos B, \qquad R_c = R/2\cos C.$$

Since these circles intersect in pairs in A, B, C and meet at the circumcenter O of triangle ABC, O is the incenter of triangle $O_aO_bO_c$.† Triangle $O_aO_bO_c$ is directly homothetic to the orthic triangle $A'B'C'$. Since O and H are corresponding points of $O_aO_bO_c$ and $A'B'C'$, the lines O_aA', O_bB', O_cC' are concurrent in a point P of the Euler line of ABC, and

$$(1) \qquad k = PH/PO = r'/\tfrac{1}{2}R = 2R\cos A\cos B\cos C/\tfrac{1}{2}R = 4\cos A\cos B\cos C,$$

$r'=2R\cos A\cos B\cos C$ where $R/2$ is the inradius of triangles $A'B'C'$ and $O_aO_bO_c$. Thus we have

THEOREM 1. *Triangle $O_aO_bO_c$ is the transform, under the homothety (P, k), of the orthic triangle $A'B'C'$, where P is the point which divides the segment HO in the ratio*

$$k = 4\cos A\cos B\cos C.$$

The normal coördinates of P with respect to ABC are $\cos 2A/\cos A$, $\cos 2B/\cos B$, $\cos 2C/\cos C$. It is not difficult to verify that P is the center of similitude of triangles $O_aO_bO_c$ and $A'B'C'$.

3. **The twin point of the circumcenter of the reference triangle.** The circles (O_a'), (O_b'), (O_c'), (with centers O_a', O_b', O_c'), symmetric to the circles (O_a), (O_b), (O_c), with respect to BC, CA, AB, intersect in the twin point [D1] Q of O. These circles intersect in pairs at A, B, C, so that O and Q are the foci of a conic Σ inscribed in the triangle $O_a'O_b'O_c'$. The axis of Σ along OQ has a length equal to R. The locus of points symmetric to Q with respect to the tangent lines

* Translated from the French by Col. W. E. Byrne.

† If A is a right angle, O_a is rejected to infinity in a direction perpendicular to BC and the midpoint O of BC is equidistant from the lines O_bO_c, O_cO_a, O_aO_b. If A is obtuse, O is the excenter of the triangle $O_aO_bO_c$ opposite vertex O_a.

of Σ is the circle (O). The points of contact of Σ with the sides of the triangle $O_a'O_b'O_c'$ are the points of intersection of the lines OA, OB, OC with the corresponding sides. The segment OQ is a diameter of the hyperbola of Jerabek [$D2$] of triangle ABC. The midpoint ω of OQ is the orthopole of the Euler line OH. Hence ω is on the circle (N) and $HQ=R$. It follows that H lies on the circle of center Q and radius R, the locus of points symmetric to O with respect to the tangent lines of Σ. Furthermore, the circle of center H and radius R contains Q and the transforms (A_2, B_2, C_2) of (A_1, B_1, C_1) and (A'', B'', C'') of (A', B', C') under the homothety $(O, 2)$. If we give to the distances from ω to the points A', B', C', which are proportional to b^2-c^2, c^2-a^2, a^2-b^2, [1], the signs of these differences, we have [2]

$$\overline{\omega A'} + \overline{\omega B'} + \overline{\omega C'} = 0$$

and, by the homothetic transformation $(O, 2)$,

$$\overline{QA''} + \overline{QB''} + \overline{QC''} = 0.$$

This gives us

THEOREM 2. *The point Q coincides with the symmetric with respect to N of the Feuerbach point of the tangential triangle of ABC* [3] *and also with the Feuerbach point of the anticomplimentary triangle* [$D3$] *of triangle $A''B''C''$ (anticomplimentation with respect to $A''B''C''$).* [4].

4. A relation between triangles $O_aO_bO_c$ and $O_a'O_b'O_c'$. The point Q', symmetrical to Q with respect to N, coincides with the point of intersection of the lines symmetric to OH with respect to the sides of ABC. [5]. The point Q is therefore the point of intersection of the lines $\Delta_a, \Delta_b, \Delta_c$, symmetric to OH with respect to the sides of $A_2B_2C_2$. The lines QA, QB, QC coincide also with $\Delta_a, \Delta_b, \Delta_c$, and as they are perpendicular to $O_b'O_c'$, $O_c'O_a'$, $O_a'O_b'$, the triangles $O_a'O_b'O_c'$ and $O_aO_bO_c$ are inversely similar. Let H_a, H_a' be the feet of the altitudes O_aH_a, $O_a'H_a'$ of triangles $O_aO_bO_c$, $O_a'O_b'O_c'$; O_a'' be symmetric to O_a' with respect to $O_b'O_c'$; A_1' be the midpoint of O_aO_a''. Since triangle $O_a''O_b'O_c'$ is symmetric to triangle $O_a'O_b'O_c'$, triangles $O_aO_bO_c$, $O_a''O_b'O_c'$, $A_1'B_1C_1$ are directly similar and the midpoint H_a'' of H_aH_a' is the foot of the altitude $A_1'H_a''$ of triangle $A_1'B_1C_1$. As the line $A_1'H_a'$ is parallel to O_aO_a', it is perpendicular to B_1C_1, and consequently it coincides with $A_1'H_a''$. Hence $H_a'H_a$ is parallel to O_aO_a' and, if we denote by H' the intersection of O_aH_a and $O_a'H_a'$, we have

$$O_aH'/H'H_a = O_a'H'/H'H_a'.$$

Thus the point H', the intersection of any two corresponding altitudes, is the double point of the inversely conformal collineation carrying $O_aO_bO_c$ into $O_a'O_b'O_c'$; H' is the common orthocenter of triangles $O_aO_bO_c$ and $O_a'O_b'O_c'$. We state this as

THEOREM 3. *The triangles $O_aO_bO_c$ and $O_a'O_b'O_c'$ are inversely similar, the double point being their common orthocenter.*

Note 1. If H'' is the orthocenter of triangle $O_a''O_b'O_c'$, the midpoint H_a' of $H'H''$ coincides with the orthocenter of triangle $A_1'B_1C_1$. Hence if we construct on the sides of the medial triangle $A_1B_1C_1$ of ABC triangles directly similar to triangle $O_aO_bO_c$ $(O_a'O_b'O_c')$, their orthocenters coincide with the feet of the altitudes of the triangle $O_aO_bO_c$ $(O_a'O_b'O_c')$.

Note 2. The properties discussed in this section also hold when the center O of the circle (O) is replaced by an arbitrary point of the plane of the reference triangle.

5. **A relation between the orthic triangle and triangle $O_a'O_b'O_c'$.** In virtue of (1), the area of triangle $O_aO_bO_c$ has for its value*

$$\overline{O_aO_bO_c} = S/8 \cos A \cos B \cos C.$$

In magnitude and sign [6] we have the relation

$$\overline{O_aO_bO_c} + \overline{O_a'O_b'O_c'} = \overline{ABC} = S$$

so that

$$\overline{O_a'O_b'O_c'} = S(1 - 1/8 \cos A \cos B \cos C),$$

and

$$(2) \qquad k'' = O_bO_c/O_b'O_c' = [1/(1 - 8 \cos A \cos B \cos C)]^{1/2} = R/OH.$$

Furthermore

$$O_bO_c = R \sin A/2 \cos B \cos C, \qquad O_b'O_c' = OH \sin A/2 \cos B \cos C,$$
$$O_cO_a = R \sin B/2 \cos C \cos A, \qquad O_c'O_a' = OH \sin B/2 \cos C \cos A,$$
$$O_aO_b = R \sin C/2 \cos A \cos B, \qquad O_a'O_b' = OH \sin C/2 \cos A \cos B,$$

from which we obtain the ratio of similitude

$$k''' = O_b'O_c'/B'C' = OH/4R \cos A \cos B \cos C$$

of triangles $O_a'O_b'O_c'$ and $A'B'C'$. If we orthogonally project the orthocenter H on OO_a', simple calculations give

$$HO_a' = OH/2 \cos A, \qquad HO_b' = OH/2 \cos B, \qquad HO_c' = OH/2 \cos C.$$

From this we obtain the ratios

$$HO_a'/HA' = OH/4R \cos A \cos B \cos C = HO_b'/HB' = HO_c'/HC',$$

which show that H coincides with the double point of the inversely conformal

* The symbol $\overline{ABC}$ is used to denote the signed area of the oriented triangle ABC.

collineation carrying $O_a'O_b'O_c'$ into $A'B'C'$. As H is the incenter of triangle $A'B'C'$ and also the incenter of triangle $O_a'O_b'O_c'$, we find, by (2) that the radius of the inscribed circle of $O_a'O_b'O_c'$ is

$$\rho' = \frac{R}{2}(OH/R) = OH/2.$$

Summarizing we have

THEOREM 4. *Triangles $O_a'O_b'O_c'$ and $A'B'C'$ are inversely similar, the double point being the orthocenter H of the reference triangle, which coincides with the center of the inscribed circle, of radius $OH/2$, of the triangle $O_a'O_b'O_c'$.*

It may be verified that $O_a'O_b'O_c'$ and $A'B'C'$ (or $O_aO_bO_c$) are both metaparallel [D4] and orthologic [D5].

6. **The isogonal conjugate of the twin point Q of the circumcenter of the reference triangle.** As the points O and Q are diametrically opposite on the Jerabek hyperbola, the center of which coincides with the orthopole of OH, the isogonal conjugate Q'' of Q is on OH. The points H, Q'' divide harmonically the diameter of the circle (O) determined by OH, and $OH \cdot OQ'' = R^2$, so that $OQ'' = R/OH$. The pedal triangle $A'''B'''C'''$ of Q'' with respect to the triangle ABC is inversely similar to $A'B'C'$ and inversely homothetic to triangle $O_a'O_b'O_c'$, since the sides of these two triangles are perpendicular to AQ, BQ, CQ. Since O and Q'' are corresponding points in triangles $O_a'O_b'O_c'$ and $A'''B'''C'''$, their homothetic center T is on OH and the homothetic ratio is equal to

$$O_b'O_c'/B'''C''' = OH^2/2Rr' = 1/4 \cos A \cos B \cos C - 2.$$

We also have

$$\overline{A'B'C'}/\overline{A'''B'''C'''} = -OH^2/R^2;$$

so that

$$B'C'/B'''C''' = OH/R = O_b'O_c'/O_bO_c,$$

and finally the area relations, in magnitude and sign,

$$\overline{A'B'C'} \cdot \overline{O_aO_bO_c} = \overline{A'''B'''C'''} \cdot \overline{O_a'O_b'O_c'} = \overline{ABC}^2/4.$$

Furthermore, the center h of the incircle (h) of triangle $A'''B'''C'''$ is on OH, and

$$TH/Th = O_b'O_c'/B'''C''' = OH^2/2Rr' = OH^2/4R^2 \cos A \cos B \cos C.$$

It follows that

$$TO/TQ'' = OH^2/2Rr'$$

and that

$$hQ'' = 2Rr'/OH = 2\rho''',$$

ρ''' being the radius of circle (h).

THEOREM 5. *In a triangle ABC, the incenter h of the pedal triangle of the isogonal conjugate Q'' of the point Q, on the hyperbola of Jerabek, diametrically opposite to the circumcenter O of triangle ABC, is on the Euler line and the radius of (h) is $hQ''/2$.*

Note that between the inradii of triangles $O_aO_bO_c$, $A'B'C'$, $O_a'O_b'O_c'$ and $A'''B'''C'''$ there is the relation

$$Rr' = 2\rho'\rho'''.$$

Also the incircles (H) and (h) of triangles $O_a'O_b'O_c'$ and $A'''B'''C'''$ subtend at the points O and Q'', respectively, angles of 60°.

7. **Isogonal and harmonic conjugates.** The line AN meets the perpendicular bisector Δ of BC at A_2. Also

$$OA_2 = AH = 2R\cos A,$$

so that

$$OA_2 \cdot OO_a = R^2 = OV^2 = OV'^2,$$

where V, V' are the points of intersection of Δ and the circle (O). The lines AA_2 and AO_a are harmonic conjugates with respect to the bisectors AV, AV' of the angle A of triangle ABC. They are isogonal conjugates with respect to the angle (AO, AH) as well as with respect to the angle (AB, AC). The lines BB_2 and BO_b, CC_2 and CO_c have similar properties. Furthermore, triangles ABC and $O_aO_bO_c$ are orthologic, the orthologic centers [D5] being coincident at O.

THEOREM 6. *Triangles ABC and $O_aO_bO_c$ are both orthologic and in perspective. The center of perspectivity is the isogonal conjugate N' of the center N of the nine-point circle of triangle ABC. The orthologic centers coincide with the circumcenter O of triangle ABC, and the axis of perspectivity is normal to ON'.*

We note that N' is on the Jerabek hyperbola of triangle ABC and, by virtue of the Weill-Aybar theorem [6],

$$N'O/N'N = 2R/OH.$$

Theorem 6 is a special case of the following

THEOREM 7. *Given a triangle ABC, two isogonal conjugate points Q, Q', and an arbitrary point P of the plane of triangle ABC, the harmonic conjugates of the lines AP, BP, CP with respect to the pairs AQ and AQ', BQ and BQ', CQ and CQ', intersect in the isogonal conjugate P' of P.*

8. **A special triangle.** Let us consider the triangle ABC for which the nine-

point center N is on the circumcircle (O). For this special triangle

$$OH = 2R$$

and, consequently,

$$a^2 + b^2 + c^2 = 5R^2.$$

Some properties of this triangle have already been published [7]. Those which follow may be new.

1. The lines AO_a, BO_b, CO_c, joining the vertices of ABC to the circumcenters of triangles BOC, COA, AOB, are parallel. They are perpendicular to the straight line on which are situated the three points obtained by taking the symmetrics of A, B, C with respect to BC, CA, AB.

2. The circumcircle (Ω) of the tangential triangle of ABC coincides with the circle described on GH as a diameter.

3. The circumcircle of triangle $O_aO_bO_c$, of center G and radius $R/3$, is tangent to (O) at N.

4. The excircle (of center H) of triangle $O_a'O_b'O_c'$ is equal to the circumcircle (O) of ABC. Triangle $O_a'O_b'O_c'$ is inversely congruent to the tangential triangle of ABC.

Definitions

*D*1. Twin point. With respect to a reference triangle ABC, two points P^* and P are said to be twin points if P is the isogonal conjugate (with respect to ABC) of a point M and P^* is the isogonal conjugate (with respect to ABC) of the inverse of M with respect to the circumcircle of ABC.

In the present paper, if we take as M the orthocenter H, P will coincide with the circumcenter O of ABC and P^* will coincide with the point Q defined by Monsieur Thébault.

Also see Gallaty, *The Modern Geometry of the Triangle*, 2nd edition, page 62.

*D*2. Hyperbola of Jerabek. The hyperbola of Jerabek of a triangle ABC is the transform by isogonal conjugates with respect to ABC of the Euler line of triangle ABC.

*D*3. Anticomplimentary point. With respect to a triangle ABC of centroid G, the point P anticomplimentary to a point P of the plane of ABC is defined by

$$\overrightarrow{GP'} = -\overrightarrow{2GP}.$$

*D*4. Metaparallel triangles. Two triangles $A_1A_2A_3$, $B_1B_2B_3$ are said to be metaparallel if the lines from A_1 parallel to B_2B_3, from A_2 parallel to B_3B_1, and from A_3 parallel to B_1B_2 are concurrent. Then the parallels to A_2A_3 drawn through B_1, $\cdots$, will also be concurrent.

*D*5. Orthologic triangles. Two triangles $A_1A_2A_3$, $B_1B_2B_3$, are said to be orthologic if the perpendiculars drawn from A_1 to B_2B_3, A_2 to B_3B_1, A_3 to B_1B_2 are concurrent in a point M. Then the perpendiculars from B_1 to A_2A_3, $\cdots$, will also be concurrent in a point M'. The points M and M' are called orthologic centers.

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A NAVIGATION COMPUTER

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1. **Introduction.** This paper presents a mathematical description of a computer of the circular slide rule type prepared for the graphical solution of the spherical triangle, particularly as applied to the astronomical triangle. The computer is rectangular and on each side a transparent rotor is attached with the pivot point at the lower left hand corner. On each rotor appear two one-parameter families of curves, one family consisting of constant altitude curves and the other of curves of constant azimuth angle. Through each transparent rotor appears the *grid* made up of many dots. These dots define two sets of curves, one for constant meridian angle (*i.e.*, local hour angle with a range from 0° to 180°) and the other set for constant declination. The meridian angles represented are integral degrees while the declinations used are those of 44 navigation stars and a few integers under 30°.

Figure 1 shows the grid, Figure 2 the rotor, both with some intermediate curves and dots removed to permit the necessary reduction in size.

The finished computer has the following properties:

1. At one setting of a rotor both the altitude and azimuth angle are obtained.

2. The errors in altitude are rarely over four minutes of arc and their average without regard to sign is under two minutes.

3. Although, from the map projection point of view, each grid and rotor is of a size to represent $\frac{1}{8}$ of the celestial sphere, the one grid and two rotors are sufficient to solve any spherical triangle.

4. The declinations of each of 44 navigation stars are plotted permanently on the grid.

5. The entire device is about 11 by 12 inches and less than $\frac{1}{4}$ of an inch thick.

2. **A description of the mathematical principles of the computer.**

The Astronomical Triangle. The fundamental problem of finding an ob-

server's position by celestial observation is that of solving the PZS spherical triangle for h and z, given d, L and t. The vertices of this triangle are Z, the observer's zenith, P, the celestial pole above the horizon, and S, the heavenly body observed and hereafter referred to as the *star*. The arcs SP, PZ, and ZS are respectively the complements of the star's declination, the observer's assumed latitude, and the star's altitude. Given the quantities d, L, and t, the

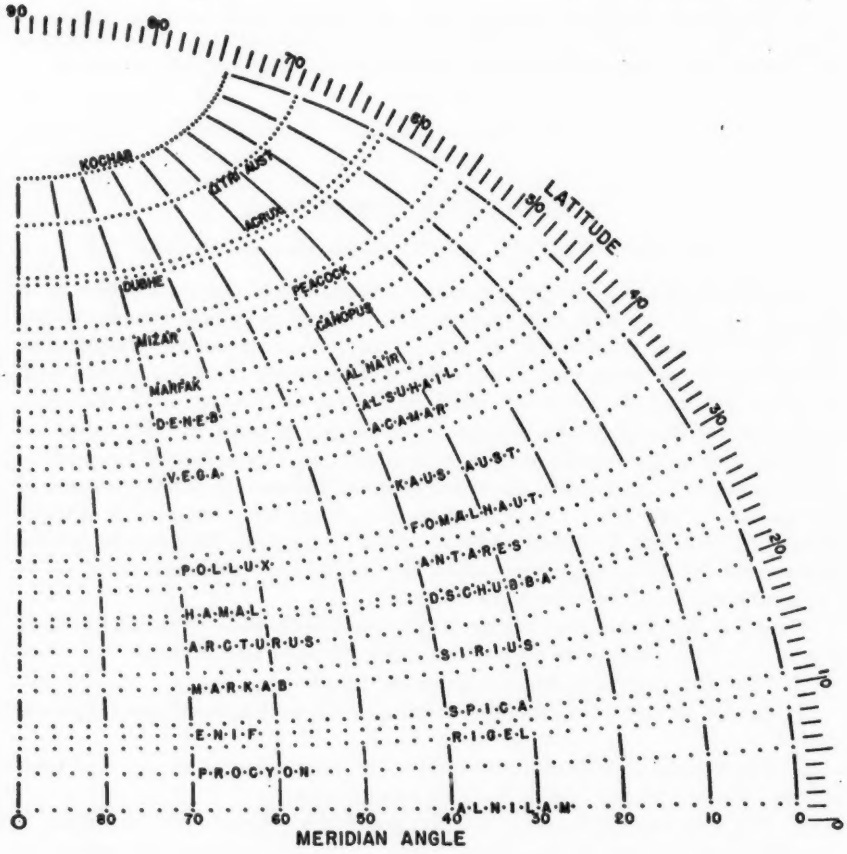


Fig. 1

quantities h and z are determined. They may be computed in any numerical case by using the formulas of spherical trigonometry or they may be found for given integral values in such tables as H.O. 214,* and H.O. 218.

* U.S. Hydrographic Office.

Graphical Solution. The required quantity, h, is expressed in terms of the known quantities d, L, and t by the law of cosines as

(1) $$\sin h = \sin d \sin L + \cos d \cos t \cos L.$$

By the substitution

(2) $$\sin d = y, \qquad \cos d \cos t = x,$$

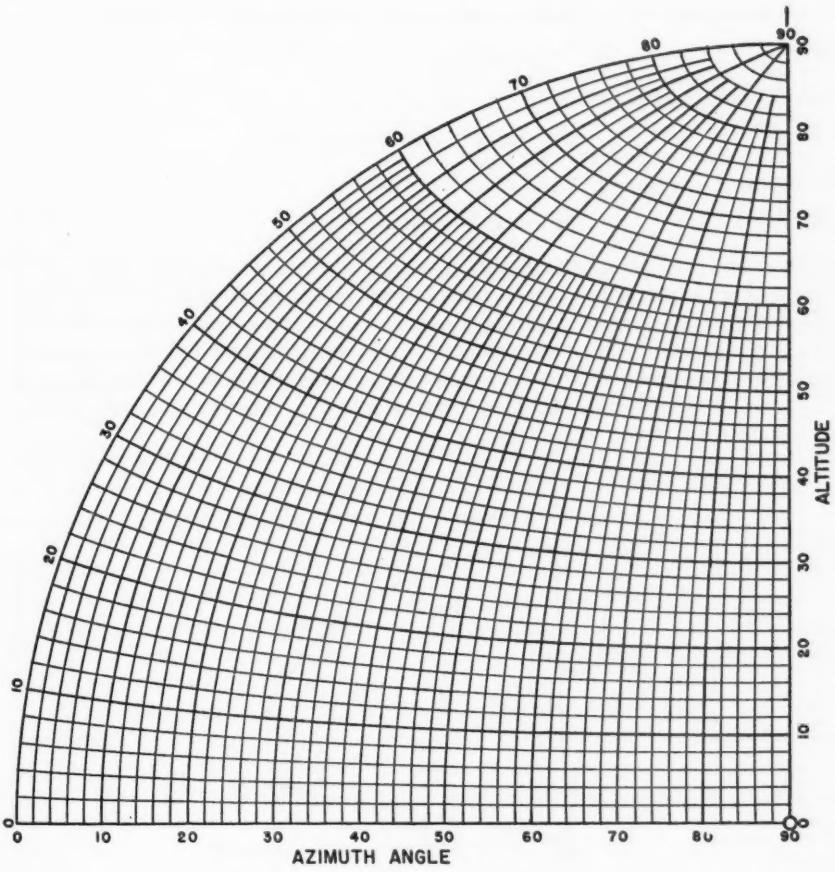


Fig. 2

Equation (1) becomes

(3) $$x \cos L + y \sin L - \sin h = 0,$$

which is the equation of a straight line with L as normal angle and $\sin h$ as normal

intercept. Equations (2) are used to define a coördinate system. It is apparent that the "d=constant curves" are straight lines parallel to the x axis while the "t=constant curves" are ellipses with common major axis. The figure, thus constructed, ultimately becomes the grid. These two families of curves are transformed to improve the scale and are replaced by dots at their points of intersection. Integer declinations are also replaced by the actual declinations of the 44 stars selected.

Applying the law of cosines again to the same triangle, we obtain:

$$\sin d = \sin h \sin L + \cos h \cos z \cos L, \tag{4}$$

which, after the substitution,

$$y' = \sin h, \qquad x' = \cos h \cos z \tag{5}$$

becomes

$$x' \cos L + y' \sin L - y = 0. \tag{6}$$

With L and y constant this equation defines another straight line with normal intercept y and the same normal angle L as that of the previous line. Obviously the coördinate system defined by Equations (5) is exactly similar to that defined by Equations (2), except that now the straight lines represent constant values of h while the ellipses correspond to constant values of z. As in the previous case, the figure thus obtained is transformed to improve the scale and becomes the rotor of the computer.

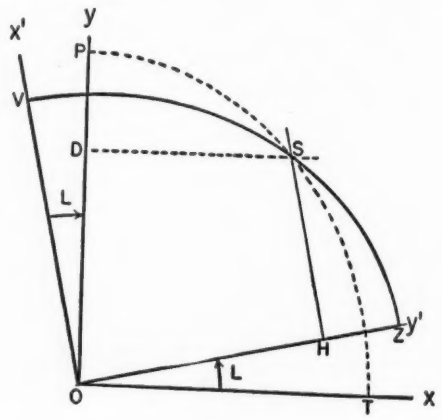


Fig. 3

Since equations (3) and (6) have the same normal angle, and the normal intercept of each is the ordinate of the other, the two may be superimposed as shown in Figure 3. The dotted ellipse PST represents one of the hour angle curves of the grid and corresponds to an assumed value of t. The dotted line SD is one of

the grid lines of constant declination to be marked on the finished computer by the name of the corresponding star. The straight line SH perpendicular to OY', and the ellipse ZSV represent, respectively, constant altitude and constant azimuth curves on the rotor. The angle L of the figure is the angle through which the rotor has been turned and corresponds to the assumed latitude of the problem. This is essentially the underlying principle of the computer. The star S is located on the grid from its given coördinates (t, d) on the XOY-system, the rotor is turned through the assumed latitude L and the coördinates (z, h) of the star are read on the $X'OY'$-system.

The Transformation. As the reader will observe, the lines and ellipses described previously in connection with the XOY-system make up an orthographic projection of the hour circles and parallels of the celestial sphere. The corresponding curves of the rotor represent the "parallels" of constant altitude and the vertical circles through Z. Computers of the same fundamental nature as the one described here have been built using the orthographic projection as well as some using a stereographic projection. These two projections have some advantage in preparation because of the lines, ellipses, and circles which make up their coördinate systems. The orthographic projection exhibits great radial crowding around the outside while the stereographic projection has its greatest crowding at the center. A suitable projection in our case is obtained by transforming the orthographic projection by means of the polar coördinate equations

$$\theta' = \theta, \qquad r' = \arcsin r.$$

The resulting figure is known as an *azimuthal equidistant* projection about the point O. All distances from O are true while the circumferential scale varies from the radial scale to $\pi/2$ times the radial scale.

Reduction to One Quadrant. Any quadrant of the grid figure is like any other quadrant except possibly for a reflection, and the same is true of the rotor figure. Thus one quadrant of grid figure and two quadrants of rotor figure are sufficient to represent any configuration. One quadrant of the grid figure is printed in duplicate on both sides of an opaque sheet and one transparent rotor quadrant is mounted over each.

Reference Lines of the Grid and Rotor. The grid carries rows of dots corresponding to the declination of the 44 navigation stars and a few additional rows for integer declinations under 30°. The dots on each row are spaced one degree apart, thus forming the family of hour circles. Each dot, therefore, represents the position of a star of known d and t, taken to the nearest integer. The rotor carries the set of equal altitude curves one degree apart and azimuth lines two degrees apart. The latitude scale is marked on the grid at the limiting circumference of the rotor (Figure 1).

3. Uses. The computer is primarily a navigation instrument for securing the necessary quantities (h, z) to establish the line of position. The value of t is obtained in the usual way with the assumed longitude adjusted so that t is equal to the nearest integer. Knowing the value of t and the coördinates of the star

observed, the corresponding dot of the grid is located and circled with a pencil. The rotor is turned to the assumed latitude and the value of h is read to the nearest minute by eye interpolation between the constant altitude curves. The value of z is obtained likewise and read to the nearest degree. In observing a body in the solar system the required dot is penciled in at its proper position on the grid and h and z obtained as before. The surface of the grid is matte-finished to permit marking and erasing.

The above procedure conforms to the standard method for determining the line of position. The computer may also be used for identifying a star from its approximate altitude and azimuth.

4. **Accuracy.** The device is amazingly accurate. The navigator wants the azimuth only to the nearest degree but desires the altitude to the nearest few minutes. H.O. 218 gives the altitude to the nearest minute of arc while H.O. 214 gives the altitude to the nearest tenth of a minute. Bearing in mind that one minute of arc amounts to one nautical mile and the additional fact that the sextant reading, with which the computed altitude is to be compared, is probably in error by a few minutes it appears that this computer is sufficiently accurate for position fixing in the air and in surface craft if an error of a very few miles can be tolerated. The computer was used in long over-ocean flying with results essentially as stated above.

THE JEEP PROBLEM: A MORE GENERAL SOLUTION

C. G. PHIPPS, University of Florida

1. **Introduction.** In this MONTHLY, January, 1947, N. J. Fine discusses the problem of advancing a single jeep a given distance beyond a base of supply. He finds where the dumps must be established in order that the amount of gasoline used is a minimum. This problem is one of a more general class whose solution I wish to discuss in detail here.

Now a solution which yields the minimum amount of gasoline for a given distance is also a solution which yields the maximum distance for a given initial amount, and conversely. It so happens that a solution for the latter conditions appears much easier to obtain. This is therefore the type of problem to be treated first.

Before formulating the various problems in detail, it is necessary to define the special terms that will be used. A group of jeeps traveling together will be referred to as a *caravan*. A *station* is defined as a point where it is necessary to establish a dump, or where the number of jeeps in the caravan is changed. The *home station* is the fixed base from which all jeeps originally set out. A *stage* is defined as the distance between two successive stations. In general, the dif-

ferent stages will be of different lengths. And lastly, the letter d will be used throughout to indicate the number of miles a jeep can travel on a full load of gasoline. Consequently, all distances will be given in miles.

There are possible many final dispositions of the jeeps in a caravan. Only two will be considered here: (1) a jeep may be abandoned at a station without its returning any distance, or (2) a jeep may be returned the whole distance to the home station. Other dispositions may be treated by the general methods presently to be outlined.

2. Special problems. Suppose m jeeps fully loaded with gasoline set out from the home station. Their object is to advance one of their number to the greatest possible distance away from the home station. Three cases will be considered: (*a*) none return, (*b*) all return, and (*c*) all but one return to the home station.

(*a*) *None return.* It is readily apparent that at some point gasoline must be transferred to the other jeeps and the empty ones abandoned. Otherwise the caravan would advance one of their number no farther than one jeep could advance alone. Where that transfer is made and how many are left at each station are the points to be determined.

When the caravan has advanced a distance of d/m miles, exactly one load of gasoline has been consumed. If one jeep is emptied of its load, the other $m-1$ can proceed fully loaded; the empty one can be abandoned on the spot.

If a jeep is abandoned before this distance is reached, the other jeeps can not carry all the gasoline on hand. They could proceed fully loaded as above but from a point closer to the home station. Ultimately then the farthest advance would be less than if the station were established at a distance of d/m.

On the other hand, if the now empty jeep were to be driven farther than this point, it must consume gasoline carried by the other jeeps. This procedure would correspondingly reduce the supply of gasoline available for the others. With a smaller supply, they would ultimately reach a shorter distance.

Therefore the most advantageous length for the first stage is d/m, at the end of which the first jeep is abandoned. Likewise, for the remaining $m-1$ jeeps, the second stage would be $d/(m-1)$, and so on. The last stage would be d since it would be traversed by a single jeep fully loaded. The total distance this last jeep has advanced would be d times the series

$$1 + 1/2 + 1/3 + \cdots + 1/m.$$

It has been tacitly assumed above that m jeeps partly loaded would still consume more gasoline than $m-1$ jeeps fully loaded. To simplify the solutions in the problems which follow, it is necessary in all of them to make the stronger assumption that a jeep consumes gasoline at exactly the same rate whether lightly or fully loaded.

(*b*) *All jeeps return.* In this case, it would require just as much gasoline for the return trip as for the outward trip. The caravan would advance until half a load had been used up; the other half load would be put into a dump for use on the return trip. The single jeep now empty could take enough from the

dump to return at once or it could wait until the others returned. The same amount of gasoline would be used either way.

Since half as much gasoline would be available for the outward trip, each stage would be half as long as before. Hence the greatest advance would be d times the series

$$1/2 + 1/4 + 1/6 + \cdots + 1/2m.$$

Now it would be possible to establish dumps between the stations indicated above but it would be unnecessary. There would be no saving in doing so unless a jeep used less gasoline the lighter its load, contrary to our assumption. Hereafter, therefore, we shall be concerned with only the minimum number of stations needed for the solution.

(*c*) *All jeeps but one return.* Over the first stage, there will be m trips outward bound and $m-1$ return. If the first station is established at a distance $d/(2m-1)$, this traffic will consume exactly one load of gasoline. To prove that this is the proper distance we argue as before. If the first station is established closer, not all the gasoline will be consumed. If it is established at a greater distance, gasoline will be consumed unnecessarily.

By repeating the argument, the second stage would be $d/(2m-3)$ in length, and so on. The last stage would again be d. Hence the total distance advanced by the last jeep would be d times the series

$$1 + 1/3 + 1/5 + \cdots + 1/(2m - 1).$$

(*d*) *Equivalence for one jeep.* So far we have considered a caravan of jeeps. The problem as treated by Mr. Fine involved only one jeep. We wish now to establish the equivalence between the travel of one jeep and that of m jeeps.

A single jeep can return to the home station or it can remain at the point of its greatest advance. Assume the former. Supplied with m loads of gasoline, it must shuttle on the first stage between the home station and the first dump. Each time, it departs from the home station fully loaded, deposits in a dump all but enough to return to the home station, and reaches there with an empty tank for its next load. Thus the jeep makes $2m$ trips over the first stage. The consumption of gasoline is exactly the same as if m different jeeps transported the same amount over the first stage. This is the same as case (*b*). Consequently, the minimum number of stations and their location needed to advance a single jeep to a maximum distance, and return, are identical in number and location with the stations of case (*b*).

If the jeep remains at the far end of its advance, the first stage is traversed $2m-1$ times. It should now be obvious that this situation corresponds to case (*c*).

3. **General principles of solution.** Several general principles governing the solution of this type of problem should now either be evident or else be easily established. These principles apply with equal force to the special problems above.

(I) The first of these is that, if the travel on any stage is begun with fully loaded jeeps, the length of this stage is such that all the travel along it, both outward and return, will consume exactly one load of gasoline.

Let m be the number of fully loaded jeeps (or trips of one jeep) which are outward bound. Let k be the number of jeeps which return, $0 \leqq k \leqq m$. Since there will be $m+k$ trips along this stage, the principle requires that the length of the stage be $d/(m+k)$. The proof goes the same as before: a shorter stage will not use all the gasoline; a longer stage will require too much.

It follows from this principle that, if there is to be a minimum number of stations, a stage does not end until the supply of gasoline to be transported beyond that point is an integral number of loads.

(II) Thus, if a fractional part of a jeep-load is to be consumed, it must be used on the first stage of the journey.

Or, thought of in another way, it will require an extra jeep to carry the fraction of a load. Naturally, this extra jeep will be sent back, or abandoned, as soon as possible; that is, as soon as the fractional load is used up.

(III) It is to be observed in passing that the number of terms in the series is equal to the number of loads of gasoline.

A fractional load will add an irregular term at the end. Other general principles will be developed in the section which follows.

4. **The inverse problem.** The inverse problem is that in which the distance is given and the number of stations and the amount of gasoline are each to be a minimum. Since the location of the stations for the greatest advance on a given supply of gasoline is the same as the location of the stations for going a given distance on a minimum amount, there remains only to fit the terms of the proper series to the distance to be covered in order to determine the number of loads needed.

(*a*) When the manner by which the jeeps are advanced leads us to a divergent series, as in the problems above, any distance from the home station can be reached. By Principle III, the number of loads of gasoline needed is exactly equal to the number of the terms of the series required to equal the distance. If the given distance is such that it is more than the sum of m terms of the series but less than $m+1$ terms, a fractional load must be used on the first short stage (Principle II).

(*b*) The problem of reaching a certain point has an interesting variation when it is required to arrive at the point with a given amount of gasoline.

Since the method of transporting gasoline as outlined in Section 3 enables one to reach the greatest distance on a given supply, it is the most economical method for transporting to that point the amount on hand at any moment. This amount can be shown by a table or a broken line graph or other means (not including the amount needed for the scheduled return trips). We can then select the place where the amount available, above that needed for return, is equal to the amount it is required to deliver. From that point measure back-

wards the required distance and count the number of stations included. This is the number of loads required to transport the given amount to the given distance. A fraction of an interval will require a fraction of a load for the first stage. The problem can be complicated by varying the disposition of the jeeps or the number that are to remain with the delivered gasoline.

The result above can be summarized into a general principle.

(IV) The amount of gasoline delivered unconsumed must be saved from the final stage, or stages, of the journey.

5. **Conclusions.** The conclusions reached by Mr. Fine hold here whenever the corresponding series is harmonic. The locations of the stations are the same whether the distance traveled is to be a maximum or the amount of gasoline used is to be a minimum. In addition, the number of stations established can be made a minimum, in which case their locations are unique.

The number of variations upon these problems is almost endless. One could have rendezvous points where jeeps are to assemble. One could consider the delivery of a certain number of jeeps to another supply station by having caravans meet halfway. Still another variation would be to have tank-trucks accompany the jeeps. Most of such problems can be worked by the general principles developed here.

The application of these problems is found, as Mr. Fine suggested, in polar regions or other places where there is no local supply. It is to be noted that the first solution (Section 3, Problem (*a*)) is exactly the one adopted for a space rocket with a multiple charge.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

SUBSERIES OF A MONOTONE DIVERGENT SERIES

R. W. HAMMING, Murray Hill, New Jersey

1. **Introduction.** There is a class of theorems in analysis in which the proof depends on showing that some quantity is greater than the sum of a series which is either known or proved to be divergent. In studying such theorems it is natural to ask to what extent they can be weakened by taking fewer terms. If the series is merely divergent, then at most a finite number may be neglected. If, however, the series is also monotone, then more terms may be neglected. The following theorem shows how many must be kept in order to still be able to prove theorems of this class.

THEOREM. *Let $I(n)$ be an increasing sequence of distinct positive integers, and let $\sum d_n$ be a monotone divergent series. A necessary and sufficient condition for the divergence of the subseries $\sum d_{I(n)}$ is that $I(n)/n$ be bounded.*

In the paper originally submitted the condition lim inf $I(n)/n < \infty$ was derived as an application of another theorem. For the result that the condition lim sup $I(n)/n < \infty$ is necessary, the author is indebted to R. P. Agnew.

2. **Proof of the theorem.** The proof of the sufficiency of the condition $I(n)/n < M$ is left to the reader.

To prove the necessity of the condition, we assume that

$$\limsup I(n)/n = \infty \tag{1}$$

and exhibit a monotone divergent series $\sum d_n$ for which the subseries $\sum d_{I(n)}$ converges. Let $n_0 = I(n_0) = 0$, and choose a positive integer n_1 such that the formula

$$[I(n_{k+1}) - I(n_k)]/[n_{k+1} - n_k] > 2^k \tag{2}$$

holds when $k = 0$. Then choose, in order, integers $n_2, n_3, \cdots$, such that both

$$n_{k+1} - n_k > n_k - n_{k-1} \tag{3}$$

and (2) hold for each $k = 1, 2, 3, \cdots$. For each $k = 0, 1, 2, \cdots$; choose constants $B(\nu)$ such that

$$2 \geqq B[I(n_k) + 1] \geqq B[I(n_k) + 2] \geqq \cdots \geqq B[I(n_{k+1})] \geqq 1.$$

For each $k = 0, 1, 2, \cdots$, let

$$d_\nu = B(\nu)/2^k[n_{k+1} - n_k], \qquad I(n_k) < \nu \leqq I(n_{k+1}). \tag{4}$$

Then $\sum d_n$ is a monotone decreasing series with $d_n \to 0$. But for each $k = 1, 2, 3, \cdots$, we have, where the summations cover the range

$$I(n_k) < \nu \leqq I(n_{k+1}),$$

$$\begin{aligned} \sum d_\nu &= \sum B(\nu)/2^k(n_{k+1} - n_k) > \sum 1/2^k(n_{k+1} - n_k) \\ &= [I(n_{k+1}) - I(n_k)]/[2^k(n_{k+1} - n_k)] > 1, \end{aligned}$$

and therefore $\sum d_n$ diverges. Finally, for each $k = 1, 2, 3, \cdots$, we have, where the summations cover the range,

$$n_k < \nu \leqq n_{k+1},$$

$$\sum d_{I(\nu)} = \sum B[I(\nu)]/2^k(n_{k+1} - n_k) < \sum 2/2^k(n_{k+1} - n_k) = 1/2^{k-1},$$

and therefore $\sum d_{I(n)}$ converges. Thus the series $\sum d_n$ has the required properties and the theorem is proved.

ONE MORE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

N. C. ANKENY, Stanford University

Many proofs of the Fundamental Theorem of Algebra, including various proofs based on the theory of analytic functions of a complex variable, are known. The following proof is a little different from the usual ones that use the Calculus of Residues.

To start with, we assume that the polynomial $P(z)$ is of degree $N \geqq 2$ and has real coefficients. We consider the integral

$$\int_\Gamma \frac{dz}{P(z)}, \tag{1}$$

where Γ denotes the boundary of a semi-circle: Γ consists of a straight line from $-R$ to R, where R is a large positive number, and of a semi-circle in the upper half plane, of radius R and center 0, connecting R to $-R$. When R tends to infinity, the integral over the circular arc approaches 0 by the usual argument, since the degree of $P(z)$ is at least 2.

Suppose now, in contradiction to the theorem we desire to prove, that $P(z)$ has no roots. Then $1/P(z)$ is everywhere regular and the value of (1) is 0. It follows that

$$\int_{-\infty}^{\infty} \frac{dx}{P(x)} = 0, \tag{2}$$

the integral being extended along the real axis. If, however, $P(x)$ has no roots, it keeps the same sign for all real values. This contradicts (2); and so it is proved that $P(z)$ must have roots.

If $Q(z)$ is any polynomial, let $\overline{Q}(z)$ denote the polynomial whose coefficients are conjugate complex to those of $Q(z)$ and apply the foregoing proof to $P(z) = Q(z)\overline{Q}(z)$.

CLASSROOM NOTES

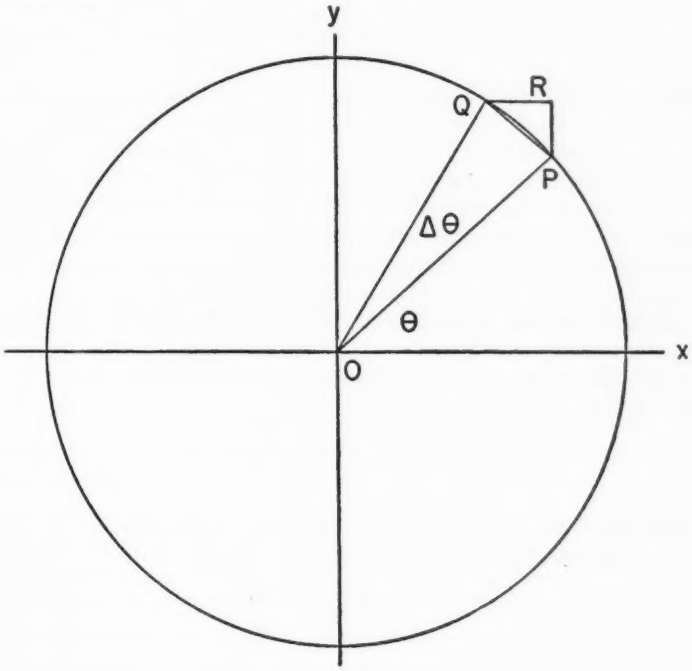
EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

DIFFERENTIATION OF THE TRIGONOMETRIC FUNCTIONS

C. I. LUBIN, University of Cincinnati

The discussion of the differentiation of the common trigonometric functions can be based on relations which do not involve a particular trigonometric function the way the usual procedure involves the sine. This unifies the development and gives it a symmetry which might be a pedagogic advantage. On the other hand, this procedure makes use of such ideas as implicit differentiation, and possibly demands a somewhat more 'sophisticated' approach than the usual treatment.



The basic relation is obtained by using the circle

(1) $$x^2 + y^2 = a^2$$

and introducing the angle, θ, made with the x-axis by a radius OP drawn to the

point $P(x, y)$ on the circumference. (See figure.) Then

$$x = a\cos\theta \qquad y = a\sin\theta. \tag{2}$$

Let the angle θ have an increment $\Delta\theta$, $\Delta\theta>0$, thereby determining the point $Q(x+\Delta x, y+\Delta y)$ lying on the circumference of the circle and in the same quadrant as P. In case P lies on one of the axes, Q is to lie in the following quadrant. We then obtain

$$\text{area of } \Delta POQ = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ x+\Delta x & y+\Delta y & 1 \end{vmatrix} = \frac{1}{2}(x\Delta y - y\Delta x) > 0.$$

The area of the sector is $\frac{1}{2}a^2\Delta\theta$ and since it is known from plane geometry that these two areas differ by an infinitesimal of higher order we can write as a limit

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = a^2 \tag{3}$$

assuming the limit $dx/d\theta$ or $dy/d\theta$ exists. This is the basic relation mentioned above. Its derivation is independent of the quadrant in which θ lies.

Recourse to the above limit theorems of plane geometry, as well as the assumption that $dx/d\theta$ or $dy/d\theta$ exists, can be avoided by considering in addition to the two areas above, the area of the quadrilateral $OPRQ$ which is greater than the area of the sector POQ. Here the point R is obtained by drawing lines parallel to the axes through P and Q and choosing the point of intersection lying outside the circle (see figure). Then on removing, say, $\Delta x/\Delta\theta$ from the expression $\frac{1}{2}(x\Delta y/\Delta\theta - y\Delta x/\Delta\theta)$ by use of equation (1), the limit $dy/d\theta$, then $dx/d\theta$, and finally (3) are readily established.

To apply this to the determination of the differentiation formulas we also need the relation

$$x\frac{dx}{d\theta} + y\frac{dy}{d\theta} = 0, \tag{4}$$

which is obtained by differentiating the equation (1) 'implicitly.'

We then find on eliminating $dx/d\theta$ from (3) and (4)

$$(x^2 + y^2)dy/d\theta = a^2x$$

or

$$dy/d\theta = x. \tag{5}$$

Similarly

$$dx/d\theta = -y.$$

These last two expressions yield the following derivatives on substituting (2)

in them:

$$\frac{d}{d\theta}(a \sin \theta) = x = a \cos \theta$$

or

$$\frac{d}{d\theta} \sin \theta = \cos \theta$$

and similarly

$$\frac{d}{d\theta} \cos \theta = -\sin \theta.$$

If we now consider θ as a function of y, namely,

$$\theta = \arcsin y/a,$$

we can rewrite (5) as

$$\frac{d\theta}{dy} = 1/x = \frac{1}{\sqrt{a^2 - y^2}} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \frac{-1}{\sqrt{a^2 - y^2}} \qquad \frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

We thus find

$$\frac{d}{dy} \arcsin y/a = \frac{1}{\sqrt{a^2 - y^2}} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \frac{-1}{\sqrt{a^2 - y^2}} \qquad \frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

The formulas for the differentiation of the remaining functions can be found in a similar manner.

Editorial Note. Another "unorthodox" derivation of these formulae is the following (essentially due to F. D. Murnaghan, this MONTHLY, vol. 53 (1946) p. 424). Consider the unit circle, and let arc length be measured positively in a counterclockwise direction from the point (1, 0). Then the angle θ (see above figure) is equal to the arc length s. But from the arc length formula:

$$(1) \qquad ds = -e\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $e = +1$ when θ is in the first or second quadrant and $e = -1$ when θ is in

the third or fourth quadrant. Now since $x=\cos\theta$ and $y=\sin\theta$, and $x^2+y^2=1$;

$$\frac{dy}{dx} = -\frac{x}{y} = -\cot\theta.$$

So from (1)

$$(2) \qquad \begin{aligned} d\theta &= -e\sqrt{1+\cot^2\theta}\, d(\cos\theta) \\ &= -e\sqrt{\csc^2\theta}\, d(\cos\theta). \end{aligned}$$

But $\sqrt{csc^2\theta}=e\ csc\ \theta$; so

$$d\theta = -e^2 \csc\theta d(\cos\theta) = -\csc\theta d(\cos\theta); \quad \text{or}$$

$$\frac{d(\cos\theta)}{d\theta} = -\sin\theta.$$

A similar argument gives the formula for the derivative of $\sin\theta$.

The chief point to notice is that in every derivation some "work" must be done by way of a limiting process. This "work" may be introduced in any of a number of places, but it can not be escaped. The objective, then, is to make this process as simple and as intuitive as possible. It is on this basis that the relative merit of various derivations should be judged. In the proof just given, advantage is taken of the customary derivation of the formula for the differential of arc length, and this enables one to get the desired result without any additional limit process. Whether or not this procedure is too "slick" is a matter which each teacher will have to decide for himself.

A TREATMENT OF BONDS BETWEEN INTEREST DATES

P. M. Hummel and C. L. Seebeck, Jr., University of Alabama

Since the flat price of a bond fluctuates according to the nearness of the forthcoming interest date, it has long been customary to quote bonds at an *and-interest price*. Consequently a purchaser pays the *quoted price* plus that part of the current bond interest payment which has already been earned. Thus if Q is the quoted price, R' the accrued bond interest, and P the flat price, then

$$(1) \qquad P = Q + R'.$$

In practice, the accrued bond interest is usually computed by the simple formula (or its equivalent)

$$(2) \qquad R' = fR,$$

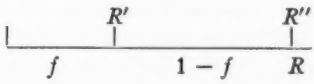
where R is the periodic bond interest payment and f is the fractional part of the interest period that has elapsed since the preceding interest date.

When a bond is bought between interest dates it is customary to consider

the purchase as consisting of two transactions, namely: (1) A loan of R' to the seller which will be repaid, without interest, out of the first bond interest payment, and (2), the purchase of a bond with a present value of Q and a first bond interest payment of $R-R'$. If C is the redemption value, an investment schedule is then set up to amortize (or accumulate) the excess, $Q-C$ (or the deficiency, $C-Q$).

The practical treatment outlined above leaves two important questions unanswered, namely: (1) If the compound interest law is used, what is the accrued bond interest? (2) Should not the investment schedule be made out for the flat price rather than the quoted price?

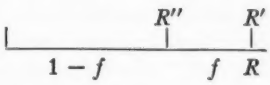
To clarify these issues we first develop a formula for R' based on the compound interest law. Clearly when the quoted price of a bond is given, the yield rate, i, which the bond will furnish is established. Consequently the forthcoming bond interest payment should be divided between the buyer and the seller in an equitable manner consistent with the compound interest law at the established rate i. To this end let R' be the seller's share, due on the date of sale, and let R'' be the buyer's share, due on the forthcoming interest date. These two shares must be equivalent to the bond interest payment due at the end of the interest period as shown on the time diagram below.



An equation of equivalence taken at the end of the interest period gives

$$R'(1+i)^{1-f}+R''=R.$$

Since both R' and R'' are unknown, we need another relation between them in order to determine their values. A second relation is obtained by symmetry. Since the determination of R' and R'' is to be equitable to both the buyer and the seller, interchanging the two times, f and $1-f$, would likewise interchange R' and R'' as shown on the following time diagram.



An equation of equivalence now gives

$$R'+R''(1+i)^{f}=R.$$

We now solve these last two equations for R' and get

$$R'=R\frac{(1+i)^{f}-1}{i}=Rs_{\overline{f}|i}. \tag{3}$$

In a similar manner, $R''=Rs_{\overline{1-f}|i}$.

It is easily shown that $s_{\overline{f}|i}$ is always slightly less than f, so that the practical

determination of R' is slightly in favor of the seller. Also, (3) disproves the statement made in some texts, that the accrued bond interest is fR but that it is not due until the end of the period, since this statement implies that $R'=fR(1+i)^{1-f}$ which is not identical with (3).

Since (3) is an exact formula for R', it is now possible to develop an exact formula for Q. We start with the well known formula

$$P = P_0(1 + i)^f, \tag{4}$$

where P_0 is the value of the bond on the preceding interest date and may be computed from the standard bond formula

$$P_0 = C(1 + i)^{-n} + Ra_{\overline{n}|i}. \tag{5}$$

If (1), (3), (4), and (5) are combined, one readily obtains

$$Q = C(1 + i)^{-(n-f)} + Ra_{\overline{n-f}|i}. \tag{6}$$

Thus we see that the exact formula for Q is the standard bond formula (5) with n replaced by $n-f$. This means that Q is the present value, on the date of purchase, of the redemption value plus the present value of *future interest earnings* and does not include that part of the current bond interest payment which has already accrued. Consequently the total purchase price should be $Q+R'$. Since (5) and (6) have the same form, it can be said that (5) is a formula for the quoted price of a bond which is valid whether n, the number of interest periods until maturity, is an integer or not.

In actual practice, Q is usually given whereas only an approximate value of i is known. Consequently the exact formulas developed here are of little practical value. However, these exact formulas justify the common accounting practice of treating Q and R' as separate transactions. Therefore when the buyer of a bond considers R' as a temporary loan which will be repaid out of the first bond interest payment, and treats Q as the book value of the bond that is being purchased and consequently sets up an investment schedule with Q as the original value of the bond, the procedure is theoretically correct.

Finally it should be noted that the approximate formulas used in the practical treatment are obtainable from the exact formulas developed here by replacing $(1+i)^f$ by $1+if$, and $a_{\overline{f}|i}$ by f. Since f is a fraction it can be shown that these replacements usually differ very little from the exact values.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 786. *Proposed by Michael Goldberg, Washington, D.C.*

Suppose that an equilateral triangle is circumscribed about a regular n-gon, where $n=3k\pm1$, so that one side of the n-gon lies on one of the sides of the triangle. Show that the angle subtended by this side of the n-gon at the opposite vertex of the triangle is $2\pi/3n$.

E 787. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, show that

$$s^4+(s-a)^4+(s-b)^4+(s-c)^4-a^4-b^4-c^4=12S^2,$$

where a, b, c are the sides, s the semi-perimeter, and S the area.

E 788. *Proposed by Leo Moser, University of Manitoba*

Consider a map on a spherical surface where the countries are determined by n great circles of which no three are concurrent. Show that if n is a multiple of four it is impossible to make a trip visiting each country once and only once, if travelling along a boundary or crossing at a boundary point of more than two countries is forbidden.

E 789. *Proposed by Kaidy Tan, Chip-Bee Institute, Amoy, Fukien, China*

If $y=\tan x$, show that

$$(d^ny/dx^n)\cos^{n+1}x = \begin{vmatrix} \cos x & 0 & \cdots & \sin x \\ \cos(x+\pi/2) & \cos x & \cdots & \sin(x+\pi/2) \\ \cos(x+2\pi/2) & 2\cos(x+\pi/2) & \cdots & \sin(x+2\pi/2) \\ \cdot & \cdot & \cdots & \cdot \\ \cos(x+n\pi/2) & n\cos(x+\overline{n-1}\,\pi/2) & \cdots & \sin(x+n\pi/2) \end{vmatrix}.$$

E 790. *Proposed by H. S. Wall, University of Texas*

Let

$$f(x,y)=\frac{a+xy(x^{n-2}+x^{n-3}y+x^{n-4}y^2+\cdots+y^{n-2})}{x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+y^{n-1}},$$

where n is an integer greater than unity, and $a>0$. If $x_0>0$, $x_1>0$, and x_2, x_3, x_4, $\cdots$ are computed recurrently by means of the formula

$$x_{p+2} = f(x_p, x_{p+1}), \qquad p = 0, 1, 2, \cdots$$

then

$$\lim_{p=\infty} x_p = a^{1/n}.$$

SOLUTIONS

A Conjecture by Srinivasan

E 755 [1947, 39]. *Proposed by Alfred Brauer, University of North Carolina*

Let a_1, a_2, a_3, a_4 be relatively prime integers such that

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 = 0. \tag{1}$$

Let $\alpha_\nu (\nu = 1, 2, 3, 4)$ by the smallest non-negative residue of a_ν (mod 6). Then

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \pmod 6. \tag{2}$$

In this paper, Residual Types of Partitions of 0 into Four Cubes (*The Mathematics Student*, vol. 13, 1945, pp. 47–48), A. K. Srinivasan tries to find solutions of (1) for each set of numbers α_ν satisfying (2). For instance, for $\alpha_1=\alpha_2=0$, $\alpha_3=1$, $\alpha_4=5$, he gives the solution $a_1=12$, $a_2=-54$, $a_3=19$, $a_4=53$. In the following cases he did not succeed in finding examples:

$$\begin{cases} \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 4; \\ \alpha_1 = \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 3; \\ \alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 3, \quad \alpha_4 = 5; \end{cases} \tag{3}$$

and in the cases obtained from (3) if each α_ν is replaced by $6-\alpha_\nu$. He conjectures that these cases are impossible. Prove that this conjecture is true.

Solution by the Proposer. Let

$$a_\nu = 6q_\nu + \alpha_\nu \qquad (\nu = 1, 2, 3, 4).$$

Then

$$a_\nu^3 = 216q_\nu^3 + 108q_\nu^2\alpha_\nu + 18q_\nu\alpha_\nu^2 + \alpha_\nu^3 \equiv \alpha_\nu^3 \pmod 9.$$

Now

$$\alpha_\nu^3 \equiv 0 \pmod 9 \quad \text{if} \quad \alpha_\nu \equiv 0 \pmod 3,$$

$$\alpha_\nu^3 \equiv 1 \pmod 9 \quad \text{if} \quad \alpha_\nu \equiv 1 \pmod 3,$$

$$\alpha_\nu^3 \equiv 8 \pmod 9 \quad \text{if} \quad \alpha_\nu \equiv 2 \pmod 3.$$

Since

$$\sum_{r=1}^{4} \alpha_r^3 \equiv \sum_{r=1}^{4} a_r^3 \equiv 0 \pmod 9,$$

the cases given in the problem are impossible, because in these cases

$$\sum_{r=1}^{4} \alpha_r^3 \equiv 3 \text{ or } 6 \pmod 9.$$

Also solved by R. A. Bradley and D. H. Browne. Browne employed the general three-parameter solution of (1) as given by Euler and later simplified by Binet. The Euler-Binet formulas may be found in Carmichael, *Diophantine Analysis* (1915), pp. 62–66, or in Hardy and Wright, *An Introduction to the Theory of Numbers* (1938), pp. 198–203.

Expansion of a Determinant

E 756 [1947, 107]. *Proposed by G. Pólya, Stanford University*

Show that

$$\begin{vmatrix} a-x & 1 & 0 & 0 & \cdots & 0 \\ \binom{a}{2} & a-x & 1 & 0 & \cdots & 0 \\ \binom{a}{3} & \binom{a}{2} & a-x & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \binom{a}{n} & \binom{a}{n-1} & \binom{a}{n-2} & \binom{a}{n-3} & \cdots & a-x \end{vmatrix}$$

$$= \binom{a+n-1}{n} - \binom{2a+n-2}{n-1}x + \binom{3a+n-3}{n-2}x^2 - \cdots + (-1)^n x^n.$$

Solution by the Proposer. 1. We consider first the general problem: Being given $a_0, a_1, a_2, \cdots$, to find $u_0, u_1, u_2, \cdots$ so that for indeterminate t

$$(a_0 + a_1 t + a_2 t^2 + \cdots)(u_0 + u_1 t + u_2 t^2 + \cdots) = 1.$$

We have

$$\begin{aligned} a_0 u_0 &= 0, \\ a_1 u_0 + a_0 u_1 &= 0, \\ a_2 u_0 + a_1 u_1 + a_0 u_2 &= 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_n u_0 + a_{n-1} u_1 + a_{n-2} u_2 + \cdots + a_0 u_n &= 0. \end{aligned}$$

Hence, by Cramer's rule,

$$(-1)^{n}a_0^{n+1}u_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots 0 \\ a_2 & a_1 & a_0 & 0 & \cdots 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots 0 \\ . & . & . & . & \cdots . \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots a_1 \end{vmatrix}.$$

2. Our problem is the special case in which

$$a_0 = 1, \qquad a_1 = a - x, \qquad a_n = \binom{a}{n} \qquad \text{for } n \geqq 2.$$

In order to evaluate the determinant we have to pick out u_n, the coefficient of t^n, in the expansion of

$$\frac{1}{(1+t)^a - xt} = \frac{(1+t)^{-a}}{1 - xt(1+t)^{-a}}$$
$$= (1+t)^{-a} + xt(1+t)^{-2a} + x^2t^2(1+t)^{-3a} + \cdots .$$

Thus

$$u_n = \binom{-a}{n} + x\binom{-2a}{n-1} + x^2\binom{-3a}{n-2} + \cdots .$$

Observing that

$$\binom{-a}{n} = \frac{-a(-a-1)\cdots(-a-n+1)}{n!} = (-1)^n\binom{a+n-1}{n},$$

we obtain the proposed equation.

Editorial Note. Many remarkable identities may be obtained from this expansion by substituting particular values for x and a, *e.g.*, $x=0$, 1, -1, a and $a=n$.

Volume of Cylindrical Wedge by Cavalieri's Theorem

E 757 [1947, 107]. *Proposed by Kirkland Stewart, College of Puget Sound*

A wedge is cut from a right circular cylinder by an oblique plane passing through a diameter of the base of the cylinder. Find the volume of the wedge using Cavalieri's theorem.

Solution by H. E. Fettis, Dayton, Ohio. Let the wedge be divided into two equal parts by a plane M through the axis of the cylinder, and let A be the area of the resulting triangular cross-section of the wedge. Construct a prism having as its base a square of side a, such that $a^2=A$, the base lying in the plane M, and having an altitude equal to the radius r of the cylinder. Cut from this prism a pyramid whose base is the base of the prism not lying in M, and whose vertex

is a point within the square A. If B and B' are the areas of cross-section of these two solids made by any plane parallel to M and at a distance x from M, then B and A are areas of similar triangles, and

$$B/A = (r^2 - x^2)/r^2.$$

Also,

$$B' = a^2 - (ax/r)^2 = a^2(1 - x^2/r^2) = A(r^2 - x^2)/r^2.$$

Therefore $B=B'$ for all values of x, so that the half wedge is, by Cavalieri's theorem, equal to the volume of the prism less the volume of the removed pyramid, which in turn equals

$$a^2r - a^2r/3 = 2a^2r/3.$$

Thus the volume of the wedge is equal to

$$4a^2r/3 = 2r^2h/3,$$

where h is the length of the longest element of the wedge.

Also solved by Elmer Latshaw and the proposer.

Editorial Note. Often, in elementary courses in solid geometry, simplicity of treatment and wider scope in application are gained by assuming Cavalieri's volume theorem. For courses where this is done the above problem makes a fine exercise.

It is well known that the volume of a sphere of radius r can very readily be obtained by Cavalieri's theorem by using for the comparison solid a right circular cylinder of radius r and altitude $2r$, with two cones removed having their bases coinciding with those of the cylinder and having common vertex at the center of the cylinder. A nice exercise is to produce, in this case, a polyhedron which may serve as the comparison solid.

The Kite

E 758 [1947, 107]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A *kite* consists of the area bounded by a major arc of a circle of radius r and the two tangents drawn at the end points of the arc. Show that (1) the area of the kite is equal to half the product of its perimeter by the radius r, (2) $Og/OG=3/2$, where g and G are the centroids of the perimeter and area, and O is the center of the kite's arc, (3) $Og'/OG'=4/3$, where g' and G' are the centroids of the surface and volume of the solid of revolution obtained by revolving the kite about its axis, (4) the plane through G' perpendicular to the axis bisects the lateral area of the solid, (5) the volume of the solid is equal to one third the product of its surface by the radius r.

Solution by the Proposer. Let A, B, M be the endpoints and the midpoint, respectively, of the arc of the kite, S the foot (where the tangents meet) of the kite, and $2(\pi-\theta)$ the central angle of the arc of the kite.

(1) We easily find

$$p = \text{perimeter} = 2r(\tan\theta + \pi - \theta),$$
$$a = \text{area} = r^2(\tan\theta + \pi - \theta)$$

whence $a = rp/2$.

(2) The perimeter p is made up of the broken line ASB and the circular arc AMB, whose lengths are $2r \tan\theta$ and $2r(\pi - \theta)$. The distances of their centroids from the diameter perpendicular to the axis of the kite are, respectively,

$$r\cos\theta + (r\sin\theta\tan\theta)/2 \quad \text{and} \quad (r\sin\theta)/(\pi - \theta).$$

Using the principle of moments we easily find

$$Og = (r\sin^3\theta)/2(\tan\theta + \pi - \theta)\cos^2\theta.$$

Similarly, the area a is made up of two triangles SAB, AOB, and the circular sector AMB with central angle $2(\pi - \theta)$. The areas of these parts are, respectively,

$$r^2\sin^3\theta/\cos\theta, \qquad r^2\sin\theta\cos\theta, \qquad r^2(\pi - \theta),$$

and the distances of their centroids from the above diameter are, respectively,

$$r(1 + 2\cos^2\theta)/3\cos\theta, \qquad (2r\cos\theta)/3, \qquad (2r\sin\theta)/3(\pi - \theta).$$

We thus find

$$OG = (r\sin^3\theta)/3(\tan\theta + \pi - \theta)\cos^2\theta,$$

whence

$$Og/OG = 3/2.$$

(3) In a manner similar to that used in (2) we find

$$Og' = (h - 2r)^2/3(h - r),$$
$$OG' = (h - 2r)^2/4(h - r),$$

where h is the height, SM, of the kite. Therefore

$$Og'/OG' = 4/3.$$

(4) The distance from O to the plane which bisects the surface of the solid and is perpendicular to the axis of the solid is calculated to be

$$x = (h - 2r)^2/4(h - r) = OG'.$$

Hence the announced property.

(5) By the usual formulas we find

$$s = \text{lateral area of solid} = \pi rh^2/(h - r),$$
$$v = \text{volume of solid} = \pi r^2h^2/3\,(h - r),$$

whence $v = rs/3$.

Note. The relations of parts (2) and (3) conform with results given by Brassine, *Journal de Liouville*, 1843, in a study of polygons circumscribed to a circle and polyhedra circumscribed to a sphere.

The Difference of Two Squares

E 759 [1947, 223]. *Proposed by Theodore Running, Ann Arbor, Michigan*

Show that $x^n-(x-a)^n$ can be expressed as the difference of two squares in at least one way for all positive integral values of n, x, and a, a less than x, not counting the obvious way when n is even.

Solution by six men from Athens, Georgia, and Columbia, South Carolina. Exercise 11, page 83, in *Elementary Number Theory*, by Uspensky and Heaslet, states that

(1) *An integer N can be represented as a difference of two squares if it is either odd or divisible by 4, otherwise not. The representation is unique if and only if N is an odd prime number.*

Now if $N=x^n-(x-a)^n$, the values n, x, $a=1$, 3, 2 yield $N=2$, which is not a difference of squares. Further, n, x, $a=2$, 2, 1 yield $N=3$, a prime which has only the "obvious" representation as a difference of squares. We conclude that the problem as stated is incorrect, and needs to be modified as follows:

(2) *If x and n are odd and $a=4k+2$, then $N=x^n-(x-a)^n$ is not a difference of squares, but if one of these fails then N is indeed a difference of squares. If n is even and greater than 2, there is at least one such expression not counting the obvious one.*

The first statement is established by canvassing the cases and observing that that $N=4k+2$ if and only if x, n are odd and $a=4k+2$. If n is even and greater than 2, N contains $x^2-(x-a)^2=a(2x-a)>1$ as a factor and is not a prime; by (1) it can be written as a difference of two squares in at least two ways and at least one must be different from the "obvious" one. The restriction $n>2$ is indeed essential since every odd prime may be expressed as a difference of two squares in one and only one way.

Remark. The probability that an example constructed at random for $n>2$ would verify the original statement is 15/16.

Also solved by Murray Barbour, Louis Berkofsky, D. H. Browne, Paul Brock, R. E. Crane, Monte Dernham, W. P. DeWitt, N. J. Fine, L. M. Kelly, C. F. Pinzka, Joseph Rosenbaum, and the proposer. A number of these solutions were not complete.

Fine proved the theorem: *Let N be an odd positive integer, $R(N)$ the number of representations $N=u^2-v^2$ $(u>0,\ v>0)$, and $d(N)$ the number of (positive) divisors of N. Then*

$$R(N) = \tfrac{1}{2}d(N), \text{ if } N \text{ is not a square,}$$
$$= \tfrac{1}{2}(d(N)-1), \text{ if } N \text{ is a square.}$$

A Faltung

E 760 [1947, 108]. *Proposed by C. D. Olds, San Jose State College*

If

$$u_k = \frac{2 \cdot 6 \cdot 10 \cdots (4k-2)}{2 \cdot 3 \cdot 4 \cdot 5 \cdots (k+1)},$$

find the value of

$$u_n + u_1 u_{n-1} + u_2 u_{n-2} + \cdots + u_{n-1} u_1 + u_n.$$

Solution by D. H. Lehmer, Berkeley, California. If we set $u_0 = 1$, then the problem is equivalent to that of finding the coefficient of x^n in the square of the function

$$y = \sum_{n=0}^{\infty} u_n x^n.$$

It is easily verified that

$$y = \{1 - (1 - 4x)^{1/2}\}/2x,$$

so that

$$y^2 = (y - 1)/x = \sum_{n=0}^{\infty} u_{n+1} x^n.$$

Hence the answer to the problem is simply u_{n+1}.

Also solved by Joshua Barlaz, Paul Brock, N. J. Fine, William Gustin, H. D. Larsen, F. C. Smith, F. Underwood, and the proposer.

Gustin pointed out that the integer u_n is the number of ways that a sequence of $n+1$ elements may be bracketed under a binary composition. For example, we have the following $u_3 = 5$ bracketing combinations of a sequence of $3+1=4$ elements: $(((\cdot\,\cdot)\cdot)\cdot)$, $((\cdot(\cdot\,\cdot))\cdot)$, $((\cdot\,\cdot)(\cdot\,\cdot))$, $(\cdot((\cdot\,\cdot)\cdot))$, $(\cdot(\cdot(\cdot\,\cdot)))$.

The problem suggested to Barlaz the associated one of finding the most general function $y = \sum_0^\infty a_n x^n$ such that $y^2 = \sum_0^\infty a_{n+1} x^n$. He easily showed that

$$y = \{1 \pm (1 - 4a_0 x)^{1/2}\}/2x.$$

The function y of the above problem is the case where we have the negative sign and $a_0 = 1$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4264. *Proposed by G. Pólya, Stanford University*

Given $a>0$, $b>0$, and given that $f(x)$ is a non-linear function such that $f(0)=0$, $f(a)=b$ and that

$$f(x) \geqq 0 \qquad f''(x) \geqq 0 \qquad\qquad 0 \leqq x \leqq a,$$

give an analytic proof that

$$2\pi \int_0^a f(x)[1 + (f'(x))^2]^{1/2} dx < \pi b(a^2 + b^2)^{1/2}.$$

(The inequality becomes intuitive when both sides are interpreted as areas of curved surfaces.)

4265. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If two tetrahedra are homothetic with respect to their common centroid, the twelve point sphere of one of these tetrahedra is tangent to the twelve point spheres of the four tetrahedra which the planes of its faces cut off from the trihedral angles of the other tetrahedron.

4266. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and a sphere (S). If the polar planes of the vertices, A, B, C, D with respect to (S) and the corresponding planes tangent to the circumsphere at A, B, C, D cut each other, respectively, on the faces BCD, CDA, DAB, ABC, the tetrahedron is orthocentric. Establish a converse theorem.

4267. *Proposed by C. F. Pinzka, Student, Rutgers University*

Let p be a prime greater than 3, and let r/ps be the sum of the harmonic series, $1+1/2+1/3+\cdots$, to p terms. Prove that p^3 divides $r-s$.

4268. *Proposed by Paul Erdös, Syracuse University*

Let $a_1<a_2<\cdots$ be an infinite sequence of integers of upper density greater than $1/k$. (Denote by $f(n)$ the number of a's up to n, then the upper density is

defined as $\overline{\lim} f(n)/n$ as $n \to \infty$.) Then for suitable t the equation

$$a_t = a_{i_1} + a_{i_2} + \cdots + a_{i_r} \qquad 1 < r < k$$

is solvable. In fact, there are infinitely many t with this property.

4269. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find all numbers $N = abcd$ of four distinct digits, zero excluded, such that the sum

$$ab + ac + ad + bc + bd + cd$$

of products of the digits two at a time shall equal the sum

$$a^2 + b^2 + c^2 + d^2$$

of the squares of the digits. For which among these is it true that the sum of the two-digit numbers ab and cd equals

$$a^2 + c^2 + d^2?$$

SOLUTIONS

A Special Sphere of the Tetrahedron

3987 [1941, 152]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The spheres S_a, S_b, S_c, S_d have their respective centers at the vertices of the tetrahedron $ABCD$, and radii whose squares are one-half the sum of the squares of the sides of the face opposite to the vertex considered; and the spheres S_a', S_b', S_c', S_d' have centers at A_1, B_1, C_1, D_1, symmetrics of A, B, C, D with respect to G the centroid of the tetrahedron, and radii A_1A, B_1B, C_1C, D_1D. Let C_a be the intersection of S_a and S_a', and similarly for C_b, C_c, C_d. Prove that: (1) The four circles C_a, C_b, C_c, C_d lie upon the same sphere Σ with its center at G and passing through the intersection of the Longchamps sphere with the anticomplementary sphere of the circumsphere of $ABCD$. (2) Show that Σ is the Monge sphere for the Steiner ellipsoid circumscribing $ABCD$.

Note. The Longchamps sphere is orthogonal to the spheres S_a, S_b, S_c, S_d See N. A. Court, *L'Enseignement Mathématique*, Geneva, 1930, pp. 31–34; V. Thébault, loc. cit., 1937, pp. 81–89. The anticomplementary sphere of the sphere $(ABCD)$ is the circumsphere of the tetrahedron formed by planes through the vertices of $ABCD$ parallel to the respective opposite faces.

Solution by the Proposer. If the lengths of the edges BC and DA, CA and DB, AB and DC are designated by a and a', b and b', c and c', then the squares of the radii of the spheres S_a, S_b, S_c, S_d, are

$$\frac{a^2 + b'^2 + c'^2}{2}, \quad \frac{b^2 + c'^2 + a'^2}{2}, \quad \frac{c^2 + a'^2 + b'^2}{2}, \quad \frac{a^2 + b^2 + c^2}{2}.$$

The radii A_1A, B_1B, C_1C, D_1D of the spheres S_a', S_b', S_c', S_d' are $3M_a/2$, $3M_b/2$, $3M_c/2$, $3M_d/2$, where M_a, M_b, M_c, M_d are the medians of the tetrahedron $ABCD$.

Let P be an arbitrary point on the intersection C_d of the spheres S_d and S_d'. The theorem concerning the square of the median, applied to the triangle DD_1P, gives

$$\overline{PD}^2 + \overline{PD_1}^2 = 2\overline{GP}^2 + \tfrac{1}{2}\overline{DD_1}^2,$$

or, by virtue of a known expression for M_d^2,*

$$\overline{GP}^2 = 3(a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/16 = \sigma^2.$$

This relation shows that the circumferences C_a, C_b, C_c, C_d are on a sphere Σ with center G and with radius σ. That this is the Monge sphere for the Steiner ellipsoid circumscribing $ABCD$ has been shown.†

Since further, the sphere Σ, the Longchamps sphere for the tetrahedron $ABCD$ and the anticomplementary sphere of the sphere $(ABCD)$ form a pencil‡ the demonstration of the theorem is complete.

A Feuerbach Point Theorem

4059 [1942, 617]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let D, E, F be the points of contact of the inscribed circle (I) with the sides BC, CA, AB of triangle ABC, and A', B', C' the feet of its altitudes. Show that the distances of the points of intersection of the pairs of straight lines such as $B'C'$, EF from the radical axis of (I) and the nine-point circle of triangle ABC are inversely proportional to the distances of the Feuerbach point from the feet of the altitudes.

Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.§ Let ABC be taken as the reference triangle for a system of normal trilinear coördinates (x, y, z). Also let a, b, c denote the lengths of the sides BC, CA, AB; $2p = a+b+c$ the perimeter, S the area, R the radius of the circumcircle, and r the radius of the incircle.

The equations of $B'C'$ and EF are respectively

$$-x\cos A + y\cos B + z\cos C = 0, \quad -a(p-a)x + b(p-b)y + c(p-c)z = 0,$$

and the coördinates (absolute) of their intersection M are

$$x = \frac{2S(c-b)(p-a)}{bc(c-b)}, \quad y = \frac{2S(c-a)(p-b)}{bc(c-b)}, \quad z = \frac{2S(a-b)(p-c)}{bc(c-b)}.$$

* V. Thébault, this MONTHLY, 1935, p. 429.

† V. Thébault, L'Enseignement Mathématique, 1937, p. 98.

‡ V. Thébault, ibid.

§ Translated and checked by W. E. Byrne, Virginia Military Institute, Lexington, Virginia.

The required radical axis is tangent to the incircle (I) at the Feuerbach point ϕ, and its equation is given by

$$\frac{ax}{b-c}+\frac{by}{c-a}+\frac{cz}{a-b}\equiv ux+vy+wz=0.*$$

The distance from M to this line is

$$\overline{MM'}=\frac{4S(p-b)(p-c)}{bc\,|\,b-c\,|\,\sqrt{\sigma}}$$

with

$$\sigma=F(u,v,w)\equiv u^2+v^2+w^2-2vw\cos A-2wu\cos B-2uv\cos C.$$

The distance from the incenter I to this radical axis is

$$\frac{r\left|\dfrac{a}{b-c}+\dfrac{b}{c-a}+\dfrac{c}{a-b}\right|}{\sqrt{\sigma}}=r,$$

so that

$$\sqrt{\sigma}=\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}.$$

But†

$$\overline{\phi A'}=\frac{|b-c|\,(p-a)\sqrt{R}}{a\sqrt{R-2r}},$$

and therefore

$$\overline{MM'}\cdot\overline{\phi A'}=\frac{4S(p-a)(p-b)(p-c)}{abc\sqrt{\sigma(1-2r/R)}}=\frac{Sr}{\sqrt{\sigma(R^2-2rR)}},$$

which establishes the proposition.

It has been assumed throughout that $(a-b)(b-c)(c-a)\neq 0$.

An Envelope of the Third Class

4154 [1945, 220]. *Proposed by H. F. Sandham, Trinity College, Dublin*

Find the envelope of the axes of conics inscribed in a quadrilateral.

Solution by the Proposer. Let $PP'=0$, $QQ'=0$ be the tangential equations of two pairs of opposite vertices of the quadrilateral, and $IJ=0$ the equation of

* See Salmon, *Conic Sections*, §131.

† Victor Thébault, Ann. Soc. Scient. de Bruxelles, 1932, p. 3.

the circular points at infinity. The equation of any inscribed conic is then $\lambda PP'+\mu QQ'=0$. For a suitable ν, the foci of this conic are given by $\Sigma\equiv\lambda PP'+\mu QQ'+\nu IJ=0$. If r, s, t are the coördinates of the line joining the foci, then $\partial\Sigma/\partial r=\partial\Sigma/\partial s=\partial\Sigma/\partial t=0$. Hence, performing these differentiations and eliminating λ, μ, ν from the resulting three equations, we obtain

$$\begin{vmatrix} P(\partial P'/\partial r)+P'(\partial P/\partial r) & Q(\partial Q'/\partial r)+Q'(\partial Q/\partial r) & I(\partial J/\partial r)+J(\partial I/\partial r) \\ & \textit{etc.} & \\ & \textit{etc.} & \end{vmatrix}=0$$

as the required envelope.

The curve is of the third class and is the Jacobian of $PP'=QQ'=IJ=0$. It is at once verified that it touches once each the lines joining the opposite vertices, and the line at infinity.

There are three tangents to the curve from any point. One of the tangents from I is the line at infinity, so that there are just two others. These must be the axes which pass through I of conics inscribed in the quadrilateral. Since the axes of a conic which passes through a circular point coincides with the tangent there, these axes must be the tangents at I to the two inscribed conics which pass through I. Now tangents from I to these conics are pairs in an involution of which the tangents to the two conics passing through I are double lines. Since tangents from a circular point to a curve have on each of them one real point which is a real focus of the curve, the above argument shows that the two real foci of the envelope are the double points of the complex involution to which belong the foci of inscribed conics.

Hence, to sum up: The axes of conics inscribed in a quadrilateral envelop a curve of the third class which is the Jacobian of two pairs of opposite vertices and the circular points, and which touches just once the diagonals of the quadrilateral and the line at infinity. The two real finite foci are the double points of the complex involution to which belong the foci of inscribed conics.

Note. The corollary that the foci of conics inscribed in a quadrilateral are pairs in a complex involution seems to be new, though the case of the degenerate point pairs is given in Morley's *Inversive Geometry*.

Four Spheres in a Tetrahedron

4160 [1945, 281]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show how to construct four spheres passing through a given point and tangent respectively to the planes of three faces of a given tetrahedron so that the points of contact are twelve points of the same sphere.

*Solution by the Proposer.** In a tetrahedron $T\equiv ABCD$, the inscribed sphere (I, r), of center I and radius r, touches the faces BCD, CDA, DAB, ABC, at A', B', C', D'. A sphere (I, ρ), of arbitrary radius $\rho>r$, concentric to (I, r), in-

* Translation by W. E. Byrne, Virginia Military Institute.

tersects these same faces in equal circles (A'), (B'), (C'), (D'), which the rays $(A'B, A'C, A'D)$, $(B'C, B'D, B'A)$, $(C'D, C'A, C'B)$, $(D'A, D'B, D'C)$ meet in the points (X_b, X_c, X_d), (Y_c, Y_d, Y_a), (Z_d, Z_a, Z_b), (V_a, V_b, V_c). The triangles $X_bX_cX_d$, $Y_cY_dY_a$, $Z_dZ_aZ_b$, $V_aV_bV_c$ are equal since their circumscribed circles are equal and the angles $BA'C$, $CA'D$, $DA'B$ formed by the lines which join the vertices of T to the points of contact of the sphere (I, r) on the planes of the faces are the same for the four faces. [1], [2].

The planes $Y_aZ_aV_a, X_bZ_bV_b, X_cY_cV_c, X_dY_dZ_d$ form a tetrahedron $T_1 \equiv A_1B_1C_1D_1$ homothetic to the tetrahedron $T' \equiv A'B'C'D'$. The planes of the faces of T, which are perpendicular to the radii IA', IB', IC', ID' of the sphere (I, r) circumscribed about tetrahedron T', are anti-parallel to the opposite faces of T_1 in the trihedral angles (A_1), (B_1), (C_1), (D_1). The planes of the faces of T meet the edges of T_1 at the vertices of the tangential triangles $X_b'X_c'X_d'$, $Y_c'Y_d'Y_a'$, $Z_d'Z_a'Z_b'$, $V_a'V_b'V_c'$ of the equal triangles $X_bX_cX_d$, $Y_cY_dY_a$, $Z_dZ_aZ_b$, $V_aV_bV_c$. These tangential triangles are also equal. The homothetic center of the tetrahedron T and the tangential tetrahedron T_1' of T_1 is the point L whose distances from the planes of the faces of T_1' are proportional to the radii of the circles circumscribed about these faces [3], [4]. This point L, the second Lemoine point of T_1', is also the homothetic center of tetrahedrons T' and T_1 [5]. Hence with respect to tetrahedron T_1 the sphere (I, ρ) belongs to a system of Tucker spheres of axis O_1L, O_1 being the circumcenter of tetrahedron T_1.

If we construct spheres (ω_a), (ω_b), (ω_c), (ω_d), inscribed in the trihedral angles (A), (B), (C), (D) of T and tangent respectively to the three adjacent faces at (Y_a, Z_a, V_a), (X_b, Z_b, V_b), (X_c, Y_c, V_c), (X_d, Y_d, Z_d), the points A' and A_1, B' and B_1, and C' and C_1, D' and D_1 belong to the radical axes of the spheres $[(\omega_b), (\omega_c), (\omega_d)]$, $[(\omega_c), (\omega_d), (\omega_a)]$, $[(\omega_d), (\omega_a), (\omega_b)]$, $[(\omega_a), (\omega_b), (\omega_c)]$. The radical center of the spheres (ω_a), (ω_b), (ω_c), (ω_d) is therefore the point L which remains fixed when the radius ρ of the sphere (I, ρ) varies. The spheres (ω_a), (ω_b), (ω_c), (ω_d) are orthogonal to a sphere (L), of center L, which reduces to a point when $(\omega_a) \cdots$, pass through L.

Conversely, if a sphere (L) is given, the spheres (O_a), (O_b), (O_c), (O_d), orthogonal to (L) and inscribed respectively in the trihedral angles (A), (B), (C), (D), are tangent to the adjacent faces in twelve points situated on a sphere concentric to the inscribed sphere (I, r) of tetrahedron T. As a special case, when the sphere (L) is a point sphere, the spheres (ω_a''), (ω_b''), (ω_c''), (ω_d'') passing through L and having their centers on AI, BI, CI, DI, between A and I, B and I, C and I, D and I, are tangent to the four faces of T at twelve points situated on the same sphere concentric with the inscribed sphere (I, r). (Sphere of Adams.)

Note 1. We have fixed the position of the points (X_b, X_c, X_d), $\cdots$, but as the lines BA', CA', DA' meet the circle (A') in six points, there are four sets of spheres like (ω_a), (ω_b), (ω_c), (ω_d) to consider. Furthermore, to each of the pedal tetrahedrons such as T' of the eight centers of spheres tangent to the four faces of T, there corresponds a point L. There are in all eight positions of L to which there may be associated spheres such as (ω_a), (ω_b), (ω_c), (ω_d), each touching three

faces of T.

Note 2. The following remarks concerning the construction of the spheres which satisfy the problem may be added [6].

Given a tetrahedron $ABCD$ and the second point of Lemoine L, the planes drawn through L perpendicular to the lines AI, BI, CI, DI joining the vertices to the center of the inscribed sphere, cut the cones of vertices A, B, C, D circumscribed about the inscribed sphere in four circles which are all on the same sphere of center I. (Sphere of Adams.)

An isogonic tetrahedron has the lines joining the vertices of the tetrahedron to the points of contact of the opposite faces with the inscribed sphere of the tetrahedron concurrent [7]. This point of intersection coincides with the second Lemoine point L defined above. In this case we have a complete analogy between the plane figure [8] and the three-dimensional figure.

References

1. A. S. Bang, Tidsskrift for Mathematik, 1897, p. 48.
2. N. A. Court, Modern Pure Solid Geometry, Macmillan, New York, 1935. §§244, 245.
3. V. Thébault, Ann. de la Soc. Scient. de Bruxelles, 1922, p. 173.
4. R. Bouvaist, Mathesis, t. 55, pp. 352–356.
5. P. Delens, Mathesis, 1937, p. 447.
6. V. Thébault, Mathesis, t. 54, Supplément, p. 20, reference to R. Bouvaist.
7. N. A. Court, ibid, §879.
8. V. Thébault, Mathesis, t. 54, Supplément, p. 9.

A Property of the Adams Sphere of a Tetrahedron

3998 [1941, 341]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A sphere (S) is tangent to the faces of a tetrahedron $ABCD$ at the points A', B', C', D' and the straight lines AA', BB', CC', DD' are concurrent in the point P. The cones (Γ_A), (Γ_B), (Γ_C), (Γ_D) with vertices at A, B, C, D circumscribe (S). The planes through P parallel to the planes $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ cut the respective cones in four circles which lie on a sphere concentric with (S).

Solution by the Proposer. In case (S) is the inscribed sphere, the point P is the second Lemoine point mentioned in the above solution of 4160 and the proof of the present proposition is included in the final remarks there. If (S) is one of the externally tangent spheres, P is one of the associates of the second Lemoine point, and the proof follows without essential change. See also V. Thébault, *Comptes-Rendus*, 1940, p. 377; *Mathesis*, 1941, *Supplément*, p. 20; this MONTHLY, 1942, p. 170.

Upper Bound for Terms of the Binomial Expansion

4186 [1946, 45]. *Proposed by Fritz Herzog, Michigan State College*

It is known from the theory of probability that for x fixed, $0<x<1$, the largest of the $n+1$ terms ${}_nC_r x^r(1-x)^{n-r}$, $r=0, 1, \cdots, n$, is asymptotically equal to $[2\pi nx(1-x)]^{-1/2}$, as $n\to\infty$. Show that for all integral values of n and r with

$n \geqq 1$, $0 \leqq r \leqq n$ and for all values of x with $0 < x < 1$

$$ {}_nC_r x^r (1 - x)^{n-r} < 1/[2enx(1 - x)]^{1/2} $$

and that this represents the best inequality in the sense that the numerator on its right cannot be replaced by any number less than unity.

Solution by the Proposer. We put

$$ R(n, r; x) = ({}_nC_r)^2 n x^{2r+1} (1 - x)^{2n-2r+1} \tag{1} $$

and proceed to show that l.u.b. $R(n, r; x) = 1/(2e)$, the l.u.b. being taken over the ranges of n, r, and x, indicated in the problem. For fixed n and r we denote the maximum of $R(n, r; x)$ in the interval $0 < x < 1$ by $S(n, r)$ and have then to show that l.u.b. $S(n, r) = 1/2e$, this l.u.b. being taken over all integral n and r with $n \geqq 1$ and $0 \leqq r \leqq n$. By differentiation of (1) it is easily seen that the maximum of $R(n, r; x)$ as a function of x in the interval $0 < x < 1$ is obtained when $x = (2r+1)/(2n+2)$, so that

$$ S(n, r) = ({}_nC_r)^2 n (2r + 1)^{2r+1} (2n - 2r + 1)^{2n-2r+1} / (2n + 2)^{2n+2}. \tag{2} $$

We now put, for $r = 0, 1, \cdots, n-1$; $q(n, r) = S(n, r+1)/S(n, r)$ and obtain from (2)

$$ q(n, r) = \frac{(n - r)^2 (2r + 3)^{2r+3} (2n - 2r - 1)^{2n-2r-1}}{(r + 1)^2 (2r + 1)^{2r+1} (2n - 2r + 1)^{2n-2r+1}}. \tag{3} $$

Considering r for a moment as a continuous variable ($0 \leqq r \leqq n-1$), we obtain from (3)

$$ \begin{aligned} d \log q(n, r)/dr &= -2/(r + 1) + 2 \log [1 + 2/(2r + 1)] \\ &\quad - 2/(n - r) + 2 \log [1 + 2/(2n - 2r - 1)] \\ &= f(2r + 1) + f(2n - 2r - 1), \end{aligned} \tag{4} $$

where $f(t) = -4/(t+1) + 2 \log (1 + 2/t)$, $t > 0$. (Since $0 \leqq r \leqq n-1$, $2r+1 \geqq 1$ and $2n-2r-1 \geqq 1$.) From $f(t) \to 0$, as $t \to +\infty$, and $f'(t) = 4/(t+1)^2 - 4/t(t+2) < 0$, it follows that $f(t) > 0$ for $t > 0$. We thus conclude from (4) that $d \log q(n, r)/dr > 0$ and hence that $q(n, r)$ increases with r ($r = 0, 1, \cdots, n-1$). In particular, we have for $0 \leqq r < (n-1)/2$

$$ q(n, r) < q(n, n - 1 - r). \tag{5} $$

On the other hand, from (1) we have $R(n, n-r; x) = R(n, r; 1-x)$ so that $S(n, n-r) = S(n, r)$ and $q(n, n-1-r) = 1/q(n, r)$. The last relation together with (5) yields $q(n, r) < 1 < q(n, n-1-r)$ for $0 \leqq r < (n-1)/2$. (In the case n odd, $q(n, r) = 1$ for $r = (n-1)/2$.) Thus, for fixed n, $S(n, r)$ decreases with r for $0 \leqq r < (n+1)/2$ and increases with r for $(n-1)/2 < r \leqq n$. Therefore, the largest of the $S(n, r)$ ($r = 0, 1, \cdots, n$) is

$$ S(n, 0) = S(n, n) = n(2n + 1)^{2n+1}/(2n + 2)^{2n+2}, \tag{6} $$

and it remains to show that l. u. b. $S(n, 0)=1/2e$, the l. u. b. being taken over all positive integral values of n. Considering n in (6) for a moment as a continuous variable, we have $d \log S(n, 0)/dn=1/n-2 \log [1+1/(2n+1)]>1/n-2/(2n+1)>0$ (since $\log (1+u)<u$ for $u>0$). Hence $S(n, 0)$ increases with n for $n \geqq 1$. On the other hand, from (6) we have $S(n, 0)=[n/(2n+2)] [1+1/(2n+1)]^{-(2n+1)}$, whence $\lim S(n, 0)=1/2e$, as $n \to \infty$. This proves l. u. b. $S(n, 0)=1/2e$. Since $S(n, 0)$ is actually less than $1/2e$ for $n=1, 2, \cdots$, the $<$ sign in the proposed inequality (rather than $\leqq$) is justified.

The following references to the asymptotic relation from the theory of probability, mentioned in the beginning of the problem, may be of interest: E. Czuber, *Wahrscheinlichkeitsrechnung*, 3rd ed., 1914, v. I, p. 134, (7), Arne Fisher, *Mathematical Theory of Probabilities*, 1922, v. I, pp. 101–102, and H. C. Plummer, *Probability and Frequency*, 1940, p. 34, (44.1).

Greatest Integer Function

4199 [1946, 225]. *Proposed by N. J. Fine, Indianapolis, Ind.*

Let r be any integer greater than unity, n a non-negative integer, and α a non-negative integer less than r. Let ν be the number of digits in the expression of n in the scale r which are not less than $r-\alpha$, and let σ be the sum of the digits. Show that

$$\sum_{k=1}^{\infty}\left[\frac{n+\alpha r^{k-1}}{r^k}\right]=\nu+\frac{n-\sigma}{r-1},$$

where the brackets denote the greatest integer function.

Solution by R. C. Buck, Harvard University. Let $n(k)$ be the individual digits so that

$$n=\sum_{k=0}^{\infty} n(k) r^k, \qquad \sigma=\sum_{k=0}^{\infty} n(k).$$

Then,

$$\left[\frac{n+\alpha r^{k-1}}{r^k}\right]=\sum_{\lambda=k}^{\infty} n(\lambda) r^{\lambda-k}+\theta_k,$$

where

$$\theta_k=\left[r^{-k}\sum_{\lambda=1}^{k-2} n(\lambda) r^{\lambda}+\{n(k-1)+\alpha\} r^{-1}\right].$$

If $n(k-1)+\alpha<r$, then $\theta_k=0$; otherwise $\theta_k=1$. Summing from 1 to infinity on k, we have

$$\sum_{k=1}^{\infty}\left[\frac{n+\alpha r^{k-1}}{r^k}\right]=\sum_{k=1}^{\infty}\sum_{\lambda=k}^{\infty} n(\lambda) r^{\lambda-k}+\sum_{k=1}^{\infty}\theta_k$$

$$= \sum_{\lambda=1}^{\infty} n(\lambda) \sum_{k=1}^{\lambda} r^{\lambda-k} + \nu$$

$$= \sum_{\lambda=1}^{\infty} n(\lambda) \frac{r^{\lambda}-1}{r-1} + \nu$$

$$= \frac{n-\sigma}{r-1} + \nu.$$

If $\alpha=0$, this reduces to $\sum(n/r^k)=(n-\sigma)/(r-1)$. If r is a prime, this gives the highest power of p dividing $n!$

Solved also by Paul Bateman, Paul Brock, C. D. Olds, and the Proposer.

Partial Derangements

4202 [1946, 278]. *Proposed by Vladimir Karapetoff, New York, N. Y.*

In a certain game of chance, consecutive numbers from 1 to n are written on a table. The same number of discs are provided with consecutive numbers written on them. The discs are turned with the numbers down so that the players cannot see the numbers written on them. A player covers all the numbers on the table with the discs at random, because he does not see the numbers on them. The discs are then turned over and the score is made on the basis of the number of discs whose numbers agree with the numbers on the table which they are covering. It is required to deduce an expression for the chance that k of the n discs covered the correct numbers.

I. *Solution by E. S. Keeping, University of Alberta.* The number of possible arrangements of discs is $n!$ The number of arrangements with k matches and no more is A_k, where A_k is the coefficient of $x_1, x_2 \cdots x_n t^k$ in *

$$\phi = (x_1 t + x_2 + \cdots + x_n)(x_1 + x_2 t + \cdots + x_n) \cdots (x_1 + x_2 + \cdots + x_n t)$$

$$= [\sum x_i + (t-1)x_1][\sum x_i + (t-1)x_2] \cdots [\sum x_i + (t-1)x_n]$$

$$= \sum_{g=0}^{n} (\sum x_i)^g (t-1)^{n-g} [x_1 x_2 \cdots x_{n-g} + \cdots],$$

the last bracket including all the sets of $n-g$ different variables out of $x_1, x_2, \cdots, x_n$.

Now the coefficient of t^k in $(t-1)^{n-g}$ is $\binom{n-g}{k}(-1)^{n-g-k}$ and the coefficient of $x_1 x_2 \cdots x_n$ in $(\sum x_i)^g$ is $g!$ The number of ways of picking out g different variables from $x_1, x_2, \cdots, x_n$ is $\binom{n}{g}$, and for each such choice there is one and only one term in the last bracket of ϕ which will give a term $x_1 x_2 \cdots x_n$ in ϕ. Therefore the required coefficient is

$$A_k = \sum_{g=0}^{n-k} \binom{n}{g} g! \binom{n-g}{k} (-1)^{n-g-k}.$$

* Each t in such a term means that some x_j came from the jth factor $(x_1+x_2+\cdots+x_j t+\cdots+x_n)$.

The probability of k matches is therefore

$$\frac{A_k}{n!} = \sum_{g=0}^{n-k} \frac{(-1)^{n-g-k}}{k!(n-k-g)!}$$

$$= \frac{1}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-k}\frac{1}{(n-k)!}\right).$$

II. *Solution by C. F. Pinzka, Student, Rutgers University.* The k correct pairs may be chosen in $C(n, k)$ ways, the number of combinations of n objects taken k at a time. For each of these $C(n, k)$ correct pairings, there are $D(n-k)$ incorrect pairings of the remaining discs and numbers, where $D(n-k)$ is the number of derangements of $n-k$ objects. This is known (See W. W. Ball, *Mathematical Recreations and Essays*, 11th ed., pp. 46–47) to be

$$D(n-k) = (n-k)!\sum_{i=0}^{n-k}(-1)^i(1/i!).$$

The total number of ways of arranging all the discs on all the numbers is $n!$, and the required probability is therefore

$$\frac{C(n, k)D(n-k)}{n!} = \frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!}.$$

Solved also by D. W. Alling, Paul Bateman, R. C. Buck, W. B. Campbell, Clara M. Feller, and Free Jamison.

Editorial Note. See also 4146 [1946, 107–110], where many references are given. Another reference is Wilks, *Mathematical Statistics*, p. 208. See also the related problems, E 719 [1947, 45–46] and E 589 [1944, 287].

The Generalized Coin Problem

4203 [1946, 278]. *Proposed by N. J. Fine, Washington, D. C.*

This is a generalization of E 651 [1945, 42]. If one is allowed n weighings on a beam balance, what is the maximum number A_n of coins, exactly one of which is bad, from which one can isolate the bad coin and determine whether it is heavy or light? (Cf. E 712 [1946, 156].)

Solution by the Proposer. It will be shown that $A_n = (3^n-3)/2$.

I. Given k coins, exactly one of which is bad, separated into two sets X and Y (not necessarily non-empty) such that the odd coin is known to be heavy or light according as it is in X or Y; if $k \leqq 3^n$, one can isolate the bad coin and identify it as light or heavy in not more than n weighings. Put x coins of X and y coins of Y in each scale pan, where x and y are so chosen that $x+y \leqq 3^{n-1}$, $k-2x-2y \leqq 3^{n-1}$. If the scales do not balance the bad coin is among the x coins on the heavy side or the y coins on the light side. If the scales balance, the bad coin is among the unweighed coins. In either case a complete induction is possible, the statement being evidently true for $n=1$.

To show by induction that 3^n is maximum (even if one has extra good coins available) suppose that x coins of X and y coins of Y are weighed against x' coins of X, y' coins of Y and z' good coins, leaving w unweighed. The possibility of balancing (with induction) implies that $w \leqq 3^{n-1}$. The possibility of unbalance (with induction) implies that the larger of $x+y'$ and $x'+y$ does not exceed 3^{n-1}. Hence the total number cannot exceed 3^n.

II. Given $(3^n-1)/2$ coins and a single additional good coin, one can isolate and identify the bad coin in n weighings. Weigh the good coin and $(3^{n-1}-1)/2$ coins against $(3^{n-1}+1)/2$, leaving unweighed $(3^{n-1}-1)/2$. If the scales balance we have $(3^{n-1}-1)/2$ unidentified coins plus a good one, so that the induction can go through. If the scales do not balance, we have $(3^{n-1}-1)/2+(3^{n-1}+1)/2 = 3^{n-1}$ coins to which I applies.

To show that the stated number is maximum (even if more good coins are available) suppose x unidentified and y good coins are weighed against $x+y$ unidentified coins, leaving z coins unweighed. The possibility of balancing (with induction) implies that $z \leqq (3^{n-1}-1)/2$. The possibility of not balancing implies, by I, that $y+2x \leqq 3^{n-1}$. Hence the total cannot exceed $(3^{n-1}-1)/2+3^{n-1}=(3^n-1)/2$.

III. Given $(3^n-3)/2$ unidentified coins (with no extra good coins) one can isolate and identify the bad coin in n weighings. Weigh $(3^{n-1}-1)/2$ against the same number, leaving as many unweighed. If the scales balance, we have $(3^{n-1}-1)/2$ unidentified coins to work with, and in addition more than one good coin. By II only $n-1$ more weighings are needed. If the scales do not balance, we have $3^{n-1}-1$ coins to which I applies.

To prove that this number is maximum, suppose that x coins are weighed against x, leaving y unweighed. The possibility of balancing implies, by II, that $y \leqq (3^{n-1}-1)/2$. The possibility of not balancing implies, by I, that $2x \leqq 3^{n-1}$. But 3^{n-1} is odd, so that $2x \leqq 3^{n-1}-1$. Hence the total number does not exceed $(3^{n-1}-1)/2+3^{n-1}-1=(3^n-3)/2$.

Solved also by Murray Barbour, R. L. Brooks and C. A. B. Smith, H. Dowker and A. Seidenberg, Clara M. Feller, Ralph Keffer, Victor Perlo, J. Rosenbaum, and G. Szekeres.

Editorial Note. C. F. Pinzka referred to *The Mathematical Gazette*, 1945, pp. 227–229, and 1946, pp. 231–234, where two solutions are given. Rosenbaum's solution associates the individual coins with integers in the ternary scale and gives a scheme whereby the number may be obtained from the results of the weighings, similar to the second solution in the *Gazette*. Brooks and Smith give an analogous solution using vector terminology.

Every solver assumed either that the coins are marked so as to be individually identifiable at any stage, or that each scale pan is divided in some way so that it is possible to put on it coins of different sets without mixing. From the clause, "all of which appear exactly alike," in the statement of E 712, a more natural interpretation seems to be that after coins have been placed together in a scale pan they are to be considered as a single set. With this restriction the maxi-

mum seems to be given by

$$A_n = (7 \cdot 3^{n-2} - 1)/2 = 3 \cdot 3^{n-2} + 3^{n-3} + 3^{n-4} + \cdots + 3 + 1.$$

In two weighings we can determine whether the three sets of 3^{n-2} coins are of equal weight, and hence all good, or whether one set contains a light or a heavy coin. If they are all good, a third weighing (against 3^{n-3} good coins) will determine whether the set of 3^{n-3} coins is all good or contains a heavy or a light coin. Continuing thus, since one coin is known to be bad, for some s ($s=2, 3, 4, \cdots, n$) we reach in s weighings a set of 3^{n-s} coins containing a heavy or a light coin. By I in Fine's solution above, $n-s$ further weighings will isolate the bad coin. (When either X or Y is empty, the method of I comes under our present restriction.)

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Science Since 1500. By H. T. Pledge. New York, Philosophical Library, 1947. (Originally published by H. M. Stationery Office, London, 1939). 357 pages. $5.00.

Interest in the history of science has grown so rapidly in recent years that the publication of this book in America should be welcome. The reader will find it both valuable and entertaining, provided he is forewarned of its numerous shortcomings.

The style is uneven, being usually good but often so bad as to obscure the sense. Surely "the mutual ignorance of the chief scientists" (p. 55) is not idiomatic English to express their lack of personal acquaintance with one another. Spelling, punctuation and printing show similar carelessness, though much more rarely. There are about thirty inserts, in the form of illustrations, graphs, charts and maps, which are interesting in themselves but have little connection with the text.

More serious defects are connected with the author's laudable but difficult attempt to provide a unifying thread through the history of modern science. Such a thread he finds in the relation between industry and science, or as he often says, between the crafts and theory. Thus Chapter V—Mechanics, Astronomy and Optics in the 17th Century—begins

> the story of this, the most important single episode in the history of science, is a complex one. Improvements in the science of mechanics are severely tested by difficulties in astronomy, brought to the front by advances in the manipulation of lenses. Other experiments made possible by skill

in the handling of glass give definiteness to the concept of a gas. This, and advances in hydraulics, enrich the idea of the physical; and the concept of mass crystallises out. With its aid, and with that of advances in mathematics, the astronomical difficulty is overcome. Doubly the advance depends upon the properties of glass.

And the whole idea is summed up in a thought-provoking sentence (p. 322) "abstract science is not knowledge of nature but of the artefacts of industry from nature."

On this chosen thread the author has attempted to string a truly amazing number of facts, the indices of his book containing nearly four thousand entries. He is thereby confronted by two complementary dangers, which he does not altogether avoid. The unifying thread may not be sufficiently strong, so that facts are set down without much connection. This is certainly the case in the closing chapters, on "Real Materials." On the other hand, a desire for unity may lead an author to discern similarities where none exist. The present book is full of comparisons between mathematics and religion, psychology and bacteriology, and so forth, which have excited ridicule, *e.g.*, in the review in *Nature*, April 13, 1940. As an example, consider the following comparison (p. 324) between mathematics and geography:

we suggested in the first chapter that elbow-room rather than order came, with the Explorations, to be the dominant idea as to space. We suggested that, with its tacit implication of relativity, this led to Copericanism in the 16th century. . . . But meanwhile (in the nineteenth century) white man was being forced to realise that, on this planet, elbow-room is not unlimited: and, as if reflecting this, the idea of order rather than of room began to come to the front again, with the added point that there might be types of order more general than any sort of space. For even the abstract spaces of Frechet recognise the limitation that there must be a meaning for the "vicinity" of points.

Such an argument, apart from its lack of clarity, goes much too far in an attempt to explain the history of mathematics in terms of navigational instruments and discoverable territory. It will prove almost anything. Fréchet's spaces are based on Cantor's theory of aggregates, arising from a study of Fourier's series, which contain the trigonometrical functions used in navigation. So we have reached the opposite conclusion: it is the Portuguese explorations which are responsible for modern abstract spaces!

The author himself is uncomfortably aware that many of his linkages are fanciful and sometimes apologizes for them in the wrong places. Thus, in a paragraph on theories of heat (p. 111):

in the early 18th century, gin engulfed Britain in an overwhelming wave of drunkenness. We may, fancifully, see in these new sciences of chemistry and of heat one reaction to this! For not only is distillation one of the best methods of preparing pure substances, but distillers, Black found, were aware, as users of fuel, of a quantitative aspect of heat other than mere temperature. Two branches of technology were thus exchanging influences, at the very start, with this new theoretical science of heat. Black, in fact, ended another case of the isolation of craft knowledge and theorist's knowledge; and the union of traditions gave him the doctrine of the latent heat of freezing and of vaporisation (1757–62). This was of special interest to Watt.

But what is fanciful in a statement that the process of manufacturing gin has

contributed to our knowledge of heat and chemistry? It is true, important, and a first-class example of the author's general method.

From what has already been said, it is clear that in spite of serious flaws the book has great positive merit. Most of our quotations have been chosen to display some fault; yet they are both informative and interesting. In fact, a book which gives a perspicuous and reasonably inclusive account, in less than five lines, of the history of Cepheid variables in determining distances to the stars (p. 293), of the early enthusiasm and opposition encountered by the practice of asepsis (p. 164) and vaccination (p. 166), of the relation of logical positivism to science (p. 190), or in less than twenty lines, of the history of the insolubility of the quintic (p. 175), of the theoretical and practical discovery of Dirac's positron (p. 282), and so on almost endlessly, in such a way as to produce in the reader, through at least two-thirds of its course, a sense of mounting excitement and of pride in human achievement, such a book is worthy of respect, let its defects be what they may.

The five chapters devoted to mathematics are: IV: Mathematics before the Calculus; VI: Mathematics 1600–1800; X: Mathematical Physics (to the end of the nineteenth century); and XII and XIII: 19th-Century Mathematics. As in the rest of the book, their extreme conciseness makes it impossible to give an account of what they include. To put it roughly: everybody is mentioned at least once.

The above remarks about the book as a whole apply equally well to the mathematical chapters. It is easy to quote passages of distinct merit which contain a serious flaw. The following paragraph (p. 187) is perhaps the best example:

> another worker in algebraic numbers, Dedekind, joined hands with Weierstrass in developing a method for dealing with the position (the need for "arithmetisation" of analysis). It lay in working out an analytically usable definition of irrational numbers, by extending that of Eudoxus. Dedekind's "cut" (1858, published 1872) divided the *rational* numbers into a L(eft) and R(ight) class, each with at least one member, such that every L is less than every R. A rational number makes a cut in which it is either the greatest of the L's or the least of the R's: which both exist. An irrational is then defined as one for which neither exists, but which (Weierstrass) can be defined, and reached in practice, by a convergent infinite sequence among the L's.

Taken as a whole, the paragraph is excellent, in line with the best passages in other parts of the book. But the words "which both exist" (instead of "one *or* the other of which exists," or perhaps "both of which possibilities always exist") constitute an elementary mistake which could easily prove baffling.

It is probable that the book will be useful; but for what type of reader? Certainly not for the undergraduate, who cannot defend himself against its errors and has never heard of Fréchet. But if, as signs indicate, the history of science will soon find its place in the curriculum of the American college, then the intending lecturer will do well to turn to this book. It will provide him with a framework, and with information which he can supplement and correct from more voluminous and more exact sources.

S. H. GOULD

Advanced Mathematics for Engineers. Second Edition. By H. W. Reddick and F. H. Miller. New York, John Wiley and Sons, Inc., 1947. 12+508 pages. $5.00.

The first edition (1938) of this book was reviewed in this MONTHLY (v. 48, 204, 1941) by Professor E. B. Allen. This second edition differs but slightly in content from the first, the character of which is familiar enough to justify omission here of detailed comment. Suffice it to say that the qualified undergraduate engineering student who has finished the calculus and wishes to take a next course in mathematics pertinent to his engineering studies will find this book instructive and to his taste. The critical remarks which follow are not to be understood as detracting from the many merits of this book.

To begin with, superficially, the format will not be found as pleasing nor the quality of the paper as good as in the first edition. However, it should be emphasized that the publishers have taken pains with the book; it has been reset, legibly printed, and is singularly free from typographic errors. This edition does no disservice to the reputation of the publishers for the much admired standard of their technical publications.

All of the problems of the first edition have been retained but with considerable rearrangement within each list. A great many new exercises have been added throughout the book, and several are of interest. Others are "routine," and it is not always clear just what these and certain of the rearrangements are intended to accomplish. The answers to all the problems are given at the end of the book. The engineering student will no doubt continue to be attracted by the practical aura of the exercises; some first rate students although finding many of the exercises worthwhile for drill purposes may miss both those significantly hard and also those delicate problems by which mettle is proved.

Beside extensive revisions of the problem lists throughout the book there are certain revisions in the text proper. In Chapter I ("Ordinary Differential Equations") the same presentation occurs except for certain revisions to several sections of Article 7: (1) a numerical change in the efflux of the problem in Section *c* makes the discussion more instructive; (2) there is a literal revision of Section *d*; (3) a more careful statement of Hooke's law and subsequent discussion are given in Section *f*; (4) Figures 3 and 4 have been redrawn. In Chapter II ("Hyperbolic Functions") an article on the "geometric representation of hyperbolic functions" and accompanying figure have been deleted. In Chapter III ("Elliptic Integrals") Article 31, dealing with a mechanical brake problem, is new. Chapter IV ("Infinite Series") and Chapter VI ("Gamma and Bessel Functions") are substantially unchanged. In Chapter V ("Fourier Series") a new Section 49 called "Combination of Series" has been added; this turns out to be merely a quotation of the sine and cosine half-range series for a particular linear polynomial. In Chapter VII ("Partial Derivatives and Differential Equations") the latter parts of Articles 70 and 71 dealing respectively with the solutions of the vibrating string and one-dimensional heat-flow equations are entirely rewritten

and improved. Now (pp. 282–284) a numerical illustration of the "telephone equations" replaces the former illustration (pp. 265–267) of the "telegraph equations." Finally, the chapter is extended by the inclusion of a discussion of the partial differential equation of a vibrating membrane (Art. 77). The contents of Chapter VIII ("Vector Analysis") and Chapter IX ("Probability") remain essentially unchanged.

Chapters X and XI remain the same as in the first edition. These are intended to serve as first introductions to the study of functions of a complex variable and of the Heaviside operational calculus. A distinguishing feature in each of these chapters will be recalled by readers of the first edition. A third of the former chapter is given over to a readable introductory account of the idea of the Schwarz-Christoffel transformation, with some of the familiar applications. The second half of the last chapter may serve as an introduction, by way of purely formal procedures, to the classical Bromwich "justification" analysis.

Finally, an appendix of five pages' length has been added on "Units and Dimensional Analysis." It is perhaps convenient to have these tables of dimensions of various physical quantities collected here. The remarks on "dimensional analysis" are cursory; a few illustrations but no explicit exercises for the student are given. The table of contents and the index have been carefully revised.

In the first edition of this book there were a few rather well-chosen references to the engineering literature which were incorporated in the text. These of course remain; one can but wish that circumstances had permitted several additions to them in the years since 1938. The reader of the appendix is referred to Eshbach's handbook and Bridgman's monograph for further information on dimensional analysis; other than these only two new references to the literature have been added. The first is to a paper by I. Opatowski which appeared in this MONTHLY (v. 48, 443, 1941), the essence of which is now included as Article 31; the other reference, to some work of one of the authors, occurs in connection with an exercise in Chapter IX. It may also be remarked that it would be helpful if authors would consistently cite references to journals and books completely and in some one of the accepted ways; defects of this kind need remedying in this book.

The highways of modern engineering science are so many-laned and the traffic is so heavy and fast that the mathematical vehicles upon which we sometimes sanguinely mount our students seem to be "Model T." Let us hope that this book, and others written with the same intent, will eventually lead (with the engineer's cooperation) to courses in what really is advanced mathematics for undergraduates in engineering which will go far toward modernizing their mathematical transportation as they start off on their careers.

The appearance of the second edition of this book, implying the growing number of such courses and the wide adoption of the book itself, would seem to call for congratulations all around.

S. G. HACKER

Cours Complet de Mathématiques Élémentaires, Tome 1, Arithmétique. By J. Haag. Paris, Gauthier-Villars, 1945. 6+103 pages. 80 fr.

This is a nice little book in the theory of arithmetic from a not too advanced standpoint. The chapter headings mention the four rational operations, decimal notation, prime numbers, fractions, decimal fractions, squares and square roots, ratio and proportion, mensuration, theory of errors. There are no problems, these doubtless being provided in a supplement.

C. C. MacDuffee

NEW BOOKS RECEIVED

Applied Bessel Functions. By F. E. Relton. London and Glasgow, Blackie and Son, Ltd., 1946. 7+191 pages. 17s. 6d.

Advanced Calculus. By D. V. Widder. New York, Prentice-Hall, Inc., 1947. 16+432 pages. $5.00.

Brief Analytic Geometry. Second Edition. By T. E. Mason and C. T. Hazzard. Boston, Ginn and Co., 1947. 9+205 pages. $2.50.

Calculating Machines. By D. R. Hartree. Cambridge University Press, 1947. 40 pages. $0.75.

Curso de Matemática en Forma de Problemas. By J. Gallego-Díaz. Madrid, Dossat, 1944. 12+333 pages.

Differential and Integral Calculus. Functions of One Variable. By F. D. Murnaghan. Brooklyn, Remsen Press, 1947. 10+502 pages. $5.00.

Elements of Symbolic Logic. By H. Reichenbach. New York, The Macmillan Company, 1947. 13+444 pages. $5.00.

Great Engines and Great Planes. By W. W. Stout. Detroit, Chrysler Corporation, 1947. 8+133 pages.

An Introduction to Mechanics. By J. W. Campbell. New York and London, Pitman Publishing Corporation, 1947. 18+372 pages. $4.50.

Introduction to the Theory of Equations. Second Edition. By L. W. Griffiths. New York, John Wiley and Sons, Inc., 1947. 9+278 pages. $3.50.

Intermediate Algebra. Revised Edition. By H. L. Rietz, A. R. Crathorne, and L. J. Adams. New York, Henry Holt and Co., 1947. 10+294 pages. $2.40.

Mathematics as a Culture Clue and Other Essays. (Collected Works, Vol. 1). By C. J. Keyser. New York, Scripta Mathematica, 1947. 7+277 pages. $3.75.

The Physical Principles of Wave Guide Transmission and Antenna Systems. By W. H. Watson. Oxford University Press, 1947. 10+208 pages. $7.00.

Proceedings of the First Canadian Mathematical Congress, Montreal, 1945. Toronto, University of Toronto Press, 1946. 44+367 pages. $3.25.

Romping Through Mathematics. By R. W. Anderson. New York, Alfred Knopf, 1947. 150 pages. $2.50.

The Strange Story of the Quantum. By B. Hoffman. New York, Harper and Brothers, 1947. 11+239 pages. $3.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1946–47

Pi Mu Epsilon, University of Nebraska

During 1946–47, seven meetings were held by the *Nebraska Alpha* Chapter of *Pi Mu Epsilon*, including a business meeting, the initiation banquet and the annual spring picnic. The following talks were given:

The Laplace transformations, by Dale Rippe

Continuous functions which are nowhere differentiable, by Mr. W. T. Lanser

The simplest problem in the calculus of variations, by Mr. Lloyd Jackson

Pascal's triangle, by Mr. Dale M. Mesner.

Ten men were initiated at the annual banquet held December 8, while twenty-three members were initiated on May 13 at the spring picnic.

The annual Freshman and Sophomore competitions were held May 10. The Freshman prize was won by Mr. William Mundell, the Sophomore prize by Mr. William Bade.

Officers for the year 1946–47 were: President, Dick Bresee; Vice-President, Marilyn Stahl; Secretary, Louise Gardels; Treasurer, Simon Delisi.

Officers for 1947–48, who took office in March, are: President, Martha Clark; Vice-President, R. N. Scheidt; Secretary, Maurice Lamoree; Treasurer, Harold B. Frost. The faculty advisor is Miss Lulu L. Runge.

Pi Mu Epsilon, University of Alabama

The activities of the *Alabama Alpha* Chapter of *Pi Mu Epsilon* centered around applications of mathematics. Several papers showed how mathematics was used by various members of the chapter during the war. The topics were:

The Army University at Biarritz, by Dr. H. S. Thurston

Geometric proofs of some trigonometric theorems, by Dr. C. L. Seebeck, Jr.

Mathematics in meteorology, by Ria J. Clinkscales, a former U.S.N. meteorologist

Mathematics in navigation, by Herschel E. Morrison, former U. S. A. A. F. navigation instructor

Differentiating and integrating circuits, by Dr. Ferdinand H. Mitchell

Internal stresses in beams, by T. W. Wilder.

The chapter started the year with 31 members and initiated 22 new members during the year. Other activities included a party and a picnic.

The officers for 1946–47 were: Director, Emily Jones; Vice-Director, Hazel Reynolds; Treasurer, Dr. H. S. Thurston; Publicity Chairman, Linda Simpson; Secretary, Dr. C. L. Seebeck, Jr.; Social Chairman, Kathleen Cannon.

The officers elected for 1947–48 are: Director, Kathleen Cannon; Vice-Director, Linda Simpson; Treasurer, Dr. H. S. Thurston; Secretary, Dr. C. L. Seebeck, Jr.; Publicity Chairman, Dr. F. A. Lewis; Social Chairman, Susie Lee Ward.

Kappa Mu Epsilon, Central Missouri State College

The theme of the program of *Missouri Beta* Chapter of *Kappa Mu Epsilon* for the year 1946–47 was *the uses of mathematics.* The following papers were presented:

Mathematics in the social sciences, by Dr. Royal J. Briggs, Professor of Economics

Mathematics in the natural sciences, by Dr. Laura J. Nahm, Professor of Zoology

Some interesting variations of the Pythagorean theorem, by Edgar Curtis

Actuarial mathematics, by Gordon Cross

Mathematical recreations, by Mrs. Virginia Moore

Magic squares, by Ronald Evans

Radar, by James Remley.

The annual spring banquet was held on April 30. On May 7, 1947, members of the *Missouri Beta* Chapter went to the campus of William Jewell College, Liberty, Missouri, and installed the *Missouri Gamma* Chapter of *Kappa Mu Epsilon.*

The officers for the year 1946–47 were: President, Ernest Hoover; Vice-President Ronald Evans; Secretary, Robert Ellis; Treasurer, Norman Hoover.

The officers for 1947–48 will be: President, Mrs. Virginia Moore; Vice-President, Samuel Herndon; Secretary, Patricia Stewart; Treasurer, Berna Deane Rist.

Pi Mu Epsilon, University of Oklahoma

The *Oklahoma Alpha* Chapter of *Pi Mu Epsilon*, having been inactive since the summer of 1943, held a reorganization meeting in October 1946.

The following papers were presented as part of the programs of the regular meetings during the year:

Ballistic meteorology, by Mr. Otis S. Spears

Graeffe's method for the solution of algebraic equations, by Mr. Clarence R. Gates

Some aspects of topology, by Mr. Roy B. Deal

The life of Newton and the development of the calculus, by Mr. H. Milton Peek

Non-Euclidean geometry, by Mr. John D. Lennes.

The annual spring banquet was held on May 9, 1947 at which the new members were initiated. Dr. N. A. Court, Professor of Mathematics, gave the principal address, *Perplexities of a Potato-pusher.*

The chapter also sponsored a renewal of its annual all-school mathematics

contest. This contest is open to any undergraduate student and consists of problems in applied calculus and analytic geometry. A prize of ten dollars in books and a second prize of five dollars in books were given to the first and second place winners. Mr. Lester D. Fisher, junior engineer, and Mr. Kay N. Burns, sophomore engineer, were guests of honor at the spring banquet.

The officers serving for 1946–47 were: Director, Mr. Charles J. Pipes; Assistant Director, Mr. Gene Levy; Secretary-Treasurer, Mr. Garth A. Abbott.

Pi Mu Epsilon, Louisiana State University

Louisiana Alpha of *Pi Mu Epsilon* at Louisiana State University met early in October, 1946, for an organization meeting and to map out a program for the year.

The following papers were given at various meetings throughout the year:
Checking hunches, by Dr. Frank A. Rickey
A fish tale, by Dr. W. V. Parker
Intuitive discussion of the point-calculus of Grassman, by Dr. H. L. Smith
There is another way, by Dr. Paul K. Rees
The generalized water-fetching problem, by Mr. John C. Currie
The degenerate forms of conic sections, by Mr. Ernest Ikenberry.

The winner of the annual Freshman Honors Examination was Mr. Earl P. Babin, and the winner of the annual Senior Award was Mr. Charles W. McArthur.

One hundred and two new members were initiated at the May meeting.

Eight new volumes have been added to the Pi Mu Epsilon collection of books, which is housed in the Mathematics Library.

The following officers served for the year: Director, Mr. Ben Mitchell; Vice-Director, Mr. Charles McCleskey; Secretary, Miss Beverly Jean Russell; Treasurer, Miss Kathryn Jumonville; Corresponding Secretary, Dr. Houston T. Karnes.

The new officers will not be elected until the fall semester.

Mathematics Club, University of Buffalo

The activities of the club centered around lectures on different phases of mathematics given by the students at monthly meetings, the topics were:
Celestial navigation, by Charles Kurland
Modular arithmetic, by Harold Schwartz
The discovery of logarithms, by Jean Ackerman
"Math" magic, by Kathleen Butz
Contributions of the Greeks to mathematics, by Edward Fadell
Approximations of definite integrals, by Mildred Scaffidi
New numbers, by Sara Zubkoff
Calendars, by Katherine Konst
Continuous deformation of solids, by William Braun, Jr.
Probability, by Harold Schwartz.

A Christmas party was held at the December meeting at which time games were played and refreshments were served.

At the April meeting the members of the club were hosts for the annual meeting of the students from the various local high schools. The purpose of this meeting was to show prospective university students what the Mathematics Department offers.

The officers for 1946–47 were: President, Jane Noller; Vice-President, Ruth Cohen; Secretary, Shirley Schwartz; Treasurer, Robert Locke; Faculty Advisor, Dr. H. F. Montague.

The officers elected for 1947–48 are: President, Jean Ackerman; Vice-President, Edward Fadell; Secretary, Mildred Scaffidi; Treasurer, William Braun, Jr.; Faculty Advisor, Dr. Harriet F. Montague.

Kappa Mu Epsilon, Central Michigan College

The fall and spring initiations brought twenty-seven new members to the *Michigan Beta* Chapter of *Kappa Mu Epsilon*, which increased the total number of active members to fifty.

The lectures and readings presented at our monthly meetings were:

Reading of Stephen Leacock's, *The Human Element of Mathematics* and *Boarding House Geometry*, by Miss Gertrude Pratt

The fundamental importance of the theory of numbers, by Mr. Dana Sudborough

Computation of firing data for field artillery, by Thomas Selby, presented at the National *Kappa Mu Epsilon* convention, and later reprinted in the spring edition, 1947, of the *Pentagon*.

The social activities of the chapter included a sleigh ride and den party, and the annual spring picnic held at School Section Lake.

Officers for the fall semester 1946 were: President, Dorothy Michener; Vice-President, Louis Stasaski; Secretary, Raymond Williams; Treasurer, Coleen Edison; Corresponding Secretary, Dr. C. C. Richtmeyer.

Officers for the spring semester 1947 were: President, Thomas Selby; Vice-President, Leon Kimball; Secretary, Genevieve Waszkiewicz; Treasurer, Mary Welsh; Corresponding Secretary, Mr. Lester Serier.

Pi Mu Epsilon, University of Washington

Pi Mu Epsilon has begun its reorganization from inactive status. The following men have been elected to associate memberships: F. Andrews, F. Ballantine, W. J. Firey, J. L. Hildebrand, E. Schlesinger, and R. Kraft. A speech on some phase of mathematics must be delivered by an associate member before an assembled body preliminary to promotion to full membership. In conformity with this rule F. Ballantine discussed a generalization of a problem appearing in this MONTHLY entitled *The stability of balanced objects*. Mr. Kraft will give a discussion of stellar spectroscopy.

Officers pro-tem are: Director, M. Stippes; Vice-Director, R. Bradford.

NEWS AND NOTICES

EDITED BY HARRY POLLARD, Cornell University

Readers are invited to contribute to the general interest of this department by sending news items to Harry Pollard, White Hall, Cornell University, Ithaca, New York.

COMPUTING MACHINERY SECTION ANNOUNCED BY QUARTERLY

Beginning with the October issue, the quarterly journal, *Mathematical Tables and Other Aids to Computation*, will publish a new feature section, "Automatic Computing Machinery," designed to disseminate information and news on research and development in the field of high-speed automatic calculating machinery. Material should fall under the general headings of Bibliography, Technical Developments, Discussion (including correspondence), and News. Contributions to this section are invited, and should be addressed to Dr. E. W. Cannon, Head of the Mathematics Group, Machine Development Laboratory, National Bureau of Standards, Washington, D. C.

The journal, edited on behalf of the Committee on Mathematical Tables and Other Aids to Computation, is published quarterly by the National Research Council, and can be obtained from the National Academy of Sciences, 2101 Constitution Avenue, Washington, D. C. A calendar year subscription is $4.00; single numbers, $1.25. Payments should be made to the National Academy of Sciences.

AWARDS IN PRELIMINARY ACTUARIAL EXAMINATIONS

The winners of the prize awards offered by the Actuarial Society of America and the American Institute of Actuaries to the nine undergraduates ranking highest in the combined score on Part 1 and Part 2 of the 1947 Preliminary Actuarial Examinations are as follows:

First Prize of $200

James H. Chung, University of Toronto

Additional Prizes of $100

James F. A. Briggs, Yale University
George Y. Cherlin, Rutgers University
Frank H. David, Harvard University
Thomas M. Galt, University of Manitoba
Charles F. Pinzka, Rutgers University
Philip C. Rapp, University of Buffalo
Morton K. Schwartz, Brown University
James G. C. Templeton, University of Toronto

The two actuarial organizations have authorized a similar set of nine prize awards for the 1948 Examinations.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination: Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.
Part 2. General Mathematics Examination: Algebra, trigonometry, coordinate geometry, differential and integral calculus.
Part 3. Special Mathematics Examination: Finite differences, probability and statistics.

The 1948 Examinations will be administered by the College Entrance Examination Board at centers throughout the United States and Canada on May 14–15, 1948. Further information on these examinations, and an application for taking them will be found in a booklet entitled, *Preliminary Actuarial Examinations*, which may be secured from either of the following organizations:

The Actuarial Society of America
393 Seventh Avenue
New York 1, New York.

American Institute of Actuaries
135 South LaSalle Street
Chicago 3, Illinois.

COLLECTED WORKS OF G. D. BIRKHOFF

Plans for the publication of a three-volume edition of the collected mathematical papers of the late Professor George D. Birkhoff have been laid by the American Mathematical Society. It is proposed to publish the papers by the photo-offset process so as to reduce the cost and to make the papers more widely available. A tentative prepublication price of $18.00 has been set for the three volumes. Advance subscriptions payable now or later may be sent to Professor J. R. Kline, American Mathematical Society, University of Pennsylvania, Philadelphia 4, Pennsylvania.

TULANE UNIVERSITY GRADUATE PROGRAM

The graduate work in mathematics at Tulane is being expanded to incorporate a program leading to the doctorate.

PERSONAL ITEMS

The following have received Guggenheim fellowship appointments: Professor Warren Ambrose of the Massachusetts Institute of Technology; Professor Garrett Birkhoff of Harvard University; Professor P. R. Halmos of the University of Chicago; Professor Saunders MacLane of the University of Chicago; Professor A. H. Taub of the University of Washington.

The following have received National Research Council fellowship appointments in mathematics: Mr. Hing Tong of Columbia University; Mr. Herman Rubin of the University of Chicago; Dr. Daniel Zelinsky of the University of Chicago.

Professor E. J. Cartan of the University of Paris has been elected to membership in the Royal Society.

Professor Emeritus G. H. Hardy of Cambridge University has been elected to membership in the French Academy of Science.

Dr. Samuel Karlen of Princeton University has been awarded the first Harry Bateman Research Fellowship by California Institute of Technology.

Professor H. A. Kramers of the University of Leyden has been elected correspondent of the French Academy of Science.

Professor E. J. McShane of the University of Virginia has been awarded an honorary doctorate of science by Tulane University.

Dr. Irving Reiner has been appointed a member at the Institute for Advanced Study for 1947–48.

Professor J. H. Van Vleck of Harvard University has received an honorary doctorate of science from the University of Wisconsin.

Professor Hermann Weyl of the Institute for Advanced Study and Professor George Pólya of Stanford University have been elected correspondents of the French Academy of Science.

Assistant Professor Florence E. Allen of the University of Wisconsin has retired.

Dr. Harriet W. Allen of Air Reduction Company, Stamford, Connecticut, has been appointed to an associate professorship in physics at Connecticut College.

Assistant Professor Warren Ambrose of the University of Michigan has been appointed to an assistant professorship at Massachusetts Institute of Technology.

Assistant Professor B. H. Arnold of Montana State College has been appointed to an assistant professorship at Oregon State College.

Assistant Professor D. H. Ballou of Middlebury College, Middlebury, Vermont, has been promoted to an associate professorship.

Associate Professor T. A. Bancroft of the University of Georgia has accepted a position as director of the Statistical Laboratory, Alabama Polytechnic Institute.

Dr. J. D. Bankier of McMaster University, Hamilton, Ontario, has been promoted to an assistant professorship.

Assistant Professor Joseph Barnett of the Oklahoma Agricultural and Mechanical College has been promoted to an associate professorship.

Professor C. F. Barr has been made Head of the Mathematics Department at the University of Wyoming.

Dr. Grace E. Bates of Mount Holyoke College has been promoted to an assistant professorship.

Assistant Professor E. E. Betz of the United States Naval Academy has been promoted to an associate professorship.

Assistant Professor F. C. Biesale of the University of Utah has been promoted to an associate professorship.

Assistant Professor M. T. Bird of Allegheny College has been appointed to an assistant professorship at San Jose State College, San Jose, California.

Associate Professor Z. W. Birnbaum of the University of Washington has been appointed Director of the Institute of Mathematical Statistics which has been established at the University of Washington.

Associate Professor David Blackwell of Howard University has been promoted to a professorship and has been appointed Chairman of the Department of Mathematics. He is visiting Cornell University from June 10 to September 20 to work on probability.

Associate Professor A. W. Boldyreff of Wittenberg College has been appointed to an associate professorship at the University of New Mexico.

Associate Professor J. C. Brixey of the University of Oklahoma has been promoted to a professorship.

Associate Professor G. W. Brown of the Iowa State College of Agriculture and Mechanical Arts has been promoted to a professorship.

Assistant Professor R. H. Bruck of the University of Wisconsin has been promoted to an associate professorship.

Professor H. E. Buchanan of Tulane University is retiring as Chairman of the Department of Mathematics, but is returning in 1947–48 to teach.

Dr. E. L. Buell of Northwestern University has been appointed mathematician at the Aerial Measurements Laboratory, Northwestern Technical Institute, Evanston, Illinois.

Associate Professor G. P. Burns of Marshal College, Huntington, West Virginia, has accepted a position as research physicist at the Naval Research Laboratory, Washington, D. C.

Assistant Professor G. F. Carrier of Brown University has been promoted to an associate professorship.

Edmund Churchill of Rutgers University has been appointed to an assistant professorship at Antioch College.

Assistant Professor J. A. Clarkson of the University of Pennsylvania has been promoted to an associate professorship.

Professor A. B. Coble of the University of Illinois has been appointed to a professorship at Haverford College.

Dr. I. S. Cohen of the University of Pennsylvania has been promoted to an assistant professorship.

G. R. Costello has accepted a position as mathematician with the National Advisory Committee on Aeronautics, Cleveland, Ohio.

Dr. S. H. Crandall of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. D. A. Darling of the California Institute of Technology has been appointed research associate at Cornell University.

Dr. Norman Davids of Johns Hopkins University has been appointed to an associate professorship at Pennsylvania State College.

Georgia K. Del Franco of the University of Miami has been promoted to an assistant professorship.

Dr. W. W. Denton of Minden City, Michigan, has been apppointed to an

assistant professorship at the University of Arizona.

Assistant Professor J. B. Diaz of Carnegie Institute of Technology has been appointed to an assistant professorship at Brown University.

Dr. Bernard Dimsdale has accepted a position as mathematician at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Professor P. A. M. Dirac of Cambridge University has been appointed visiting professor at the Institute for Advanced Study for the current academic year.

W. W. Dolan of the University of Oklahoma has been promoted to an assistant professorship.

Associate Professor H. L. Dorwart of Washington and Jefferson College, Washington, Pennsylvania, has been promoted to a professorship.

Associate Professor T. L. Downs of the United States Naval Academy has been appointed to an associate professorship at Washington University.

Professor W. L. Duren, Jr., has been appointed Chairman of the Department of Mathematics in the College of Arts and Sciences at Tulane University.

Dr. William H. Durfee of Dartmouth College has been promoted to an assistant professorship.

Dr. Jacques Dutka of Princeton University has been appointed to an assistant professorship at Rutgers University.

M. W. Eudey of the University of California has accepted a position as operations analyst with the Army Air Force, Washington, D. C.

Assistant Professor C. J. Everett of the University of Wisconsin is on leave of absence and will be at the Los Alamos Scientific Laboratories.

Albert Furman of the University of Chicago has been appointed to an assistant professorship at Kansas State College of Agriculture and Mechanical Arts.

Professor Abel Gauthier of the University of Montreal has been appointed head of the Institute of Mathematics.

Assistant Professor Gladys Gibbens of the University of Minnesota has been promoted to an associate professorship.

Associate Professor M. E. Gillis of the University of Florida has been appointed to a professorship at Blue Mountain College, Blue Mountain, Mississippi.

Dr. M. A. Girschick of the Bureau of the Census has accepted a position as research mathematician with Douglas Aircraft Corporation, Santa Monica, California.

Associate Professor Wallace Givens of the Illinois Institute of Technology has been appointed to a professorship at the University of Tennessee.

Dr. C. H. Graves, formerly of the Office of Price Administration, has accepted a position as operations analyst with the Army Air Forces, Air Defense Command, Mitchell Field, New York.

H. J. Greenberg of Brown University has been promoted to an assistant professorship.

Bernard Greenspan of Drew University, Madison, New Jersey, has been promoted to an assistant professorship.

Associate Professor P. C. Hammer of Oregon State College has accepted a position as mathematician at the Los Alamos Scientific Laboratory.

I. H. Harris of the University of Illinois has been appointed to a professorship at Oklahoma Baptist University.

Assistant Professor Philip Hartman of Johns Hopkins University has been promoted to an associate professorship.

Assistant Professor G. E. Hay of the University of Michigan has been promoted to an associate professorship.

J. J. Hayes of the University of Utah has been promoted to an assistant professorship.

Dr. Katherine E. Hazard of the New Jersey College for Women, Rutgers University, has been promoted to an assistant professorship.

Assistant Professor Anna S. Henriques of the University of Utah has been promoted to an associate professorship.

W. C. Hoffman of the University of California at Los Angeles has accepted a position as mathematician at the Naval Electronics Laboratory, San Diego, California.

Professor Ralph Hull of the University of Nebraska has accepted a position as mathematician at Boeing Aircraft Company, Seattle, Washington.

Associate Professor Nathan Jacobson of Johns Hopkins University has been appointed to an associate professorship at Yale University.

Associate Professor E. D. Jenkins of Eastern Kentucky State Teachers College has been appointed to an associate professorship at Kent State University.

Assistant Professor Walter Jennings of Virginia Polytechnic Institute has been appointed to an assistant professorship at the Postgraduate School, United States Naval Academy.

Assistant Professor R. E. Johnson of Mount Holyoke College has been appointed to an associate professorship at Smith College.

Professor B. W. Jones of Cornell University will be on leave of absence for the current academic year and will be at California Institute of Technology.

Dr. William Karush of the University of Chicago has been promoted to an assistant professorship.

E. H. Larguier of the University of Michigan has been appointed to a professorship at Spring Hill College, Spring Hill, Alabama.

M. M. Lemme of Purdue University has been appointed to an associate professorship at Illinois Wesleyan University. He will also be assistant dean of admissions.

Assistant Professor R. J. Levit of the University of Georgia has been promoted to an associate professorship.

Associate Professor C. C. Lin of Brown University has been appointed to an associate professorship at Massachusetts Institute of Technology.

Associate Professor H. W. Linscheid of the College of Emporia, Emporia, Kansas, has been appointed to an associate professorship at the Southwestern Institute of Technology, Weatherford, Oklahoma.

A. J. Lorenz of St. Louis University has been promoted to an assistant pro-

fessorship.

R. E. Luce of Rutgers University has been promoted to an assistant professorship.

R. T. Luginbuhl of the City Bank Farmers Trust Company has been promoted to the position of actuarial assistant.

Professor J. C. C. McKinsey of Oklahoma Agricultural and Mechanical College has accepted a position with Douglas Aircraft Company, Santa Monica, California.

F. A. McMahon of Manhattan College has been promoted to an assistant professorship.

Dr. N. S. Mendelsohn of Queens University has been appointed to an assistant professorship at the University of Manitoba.

Dr. W. A. Mersman of Taylor Instrument Company, Rochester, New York, has accepted a position as mathematician with the National Advisory Committee on Aeronautics, Moffett Field, California.

Dr. D. S. Miller of Yale University has been appointed to an assistant professorship at the University of Rochester.

A. M. Mood of Iowa State College of Agriculture and Mechanical Arts has been promoted to a professorship.

Leo Moser of the University of Toronto has been appointed lecturer at the University of Manitoba.

Associate Professor Ivan Niven of Purdue University has been appointed to an associate professorship at the University of Oregon.

Dr. Anne F. O'Neill of Smith College has been promoted to an assistant professorship.

Dr. B. J. Pettis of Yale University has been appointed to an associate professorship at Tulane University of Louisiana.

Assistant Professor Everett Pitcher of Lehigh University has been promoted to an associate professorship.

Associate Professor E. J. Purcell of the University of Arizona has been promoted to a professorship.

Assistant Professor Haim Reingold of the Illinois Institute of Technology has been promoted to an associate professorship.

Assistant Professor Helene Reschovsky of Russell Sage College, Troy, New York, has been promoted to an associate professorship.

Professor C. E. Rhodes of Washington College, Chestertown, Maryland, has been appointed to a professorship at Alfred University, Alfred, New York.

Associate Professor L. F. S. Ritcey of United College, Winnipeg, Manitoba, has been appointed to a professorship in actuarial science at the University of Manitoba.

Professor H. P. Robertson of Princeton University has been appointed to a professorship at California Institute of Technology.

Dr. H. M. Schaerf of Montana State College has been appointed to an assistant professorship at Washington University.

A. E. Schild of Carnegie Institute of Technology has been promoted to an

assistant professorship.

Dr. H. M. Schwartz of the Franklin Institute has accepted a position as scientist with the Brookhaven National Laboratory, Patchogue, New York.

Assistant Professor Domina E. Spencer of Tufts College has been appointed to an assistant professorship of physics at Brown University.

Associate Professor N. E. Steenrod of the University of Michigan has been appointed to an associate professorship at Princeton University.

Professor M. H. Stone of the University of Chicago has been lecturing at the University of Brazil.

W. M. Stone of Iowa State College of Agriculture and Mechanical Arts has been appointed to an assistant professorship at Oregon State College.

Dr. Walter Strodt of Columbia University has been promoted to an assistant professorship.

F. W. Thalgott of the University of Illinois has accepted a position as mechanical engineer at the Clinton Laboratories, Oak Ridge, Tennessee.

Assistant Professor C. J. Thorne of the University of Utah has been promoted to an associate professorship.

Dr. W. J. Thron of Washington University has been promoted to an assistant professorship.

Dr. E. B. Tolsted of Brown University has been appointed to an assistant professorship at Pomona College, Claremont, California.

Assistant Professor H. C. Trimble of Iowa State Teachers College has been appointed to an associate professorship at Florida State College.

Professor G. R. Trott of Blue Mountain College, Blue Mountain, Mississippi, has been appointed to an associate professorship at the University of Mississippi.

Professor Bird M. Turner of West Virginia University has retired with the title emeritus.

Dr. G. B. Van Schaack of Union College has been appointed to an assistant professorship at Washington University, St. Louis, Missouri.

Assistant Professor Kenichi Watanabe of the University of Hawaii has been appointed assistant professor in physics at Wabash College.

Associate Professor F. P. Welch of Mississippi State College has been appointed to a professorship at Washington and Lee University.

R. L. Wine of the University of Oklahoma has been appointed to an assistant professorship at Washington and Lee University.

Dr. Y. C. Wong of the University of Pennsylvania has been appointed to a professorship at the National Sun Yat-Sen University, Canton, China.

Assistant Professor G. S. Young of Purdue University has been appointed to an assistant professorship at the University of Michigan.

Professor Oscar Zariski of the University of Illinois has been appointed to a professorship at Harvard University.

The following appointments to instructorships are announced:

Allegheny College: E. A. Sturley

Brown University: Dr. F. M. Stewart
Grinnell College: R. E. Cross
Harvard University: Dr. W. J. Leveque, Dr. G. R. MacLane
Haverford College: D. L. Thomsen
Rutgers University: A. G. Anderson, A. F. Bartholomay, Dr. Erwin Biser, Dr. R. M. Cohn, A. W. Goodman
Syracuse University: Dr. Erik Hemmingsen
Texas Technological College: Mrs. Virginia B. Roberts
Washington University: Dr. C. W. Mathews
Western Illinois State Technological College, Macomb, Ill.: J. J. Stipanowich
University of California at Los Angeles: L. J. Paige, J. D. Swift
University of Minnesota: Irwin Stoner
University of Pennsylvania: D. M. Adelman
University of Tennessee: R. L. Wilson
University of Wisconsin: Dr. A. M. Mark
University of Wisconsin at Racine: M. R. Moore

Dean Arthur Léveillé, who was Head of the Institute of Mathematics at the University of Montreal, died March 15, 1947.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-seventh regular meeting of the Southern California Section of the Mathematical Association of America was held at Pomona College, Claremont, California, on Saturday, March 8, 1947. Professor H. J. Hamilton, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was one hundred and thirty, including the following forty-nine members of the Association: L. J. Adams, O. W. Albert, E. F. Beckenbach, Clifford Bell, Garrett Birkhoff, L. T. Black, H. F. Bohnenblust, Frances L. Campbell, L. M. Coffin, F. E. Cothran, D. R. Curtis, P. H. Daus, R. P. Dilworth, H. P. Edmundson, Iva B. Ernsberger, C. M. Fulton, J. W. Green, W. S. Gustin, H. J. Hamilton, R. B. Herrera, R. E. Horton, D. H. Hyers, C. G. Jaeger, Glenn James, G. R. Kaelin, Margaret B. Lehman, Ada A. McClellan, G. F. McEwen, P. M. Niersbach, R. P. Peterson, W. T. Puckett, H. R. Pyle, L. T. Ratner, E. C. Rex, G. E. F. Sherwood, I. S. Sokolnikoff, R. H. Sorgenfrey, D. V. Steed, H. L. Steinberg, A. E. Taylor, F. B. Thompson, W. I. Thompson, S. E. Urner, F. A. Valentine, Morgan Ward, R. L. White, Mabel G. Whiting, Euphemia R. Worthington.

At the business meeting the following officers were elected for the next academic year: Chairman, D. V. Steed, University of Southern California;

Vice-Chairman, E. F. Beckenbach, University of California at Los Angeles; Program Committee, J. W. Green (Chairman), O. W. Albert, C. W. Trigg, and the Secretary, Paul H. Daus, ex-officio. The regular meeting was scheduled for March 13, 1948, at the University of Redlands.

The following papers were presented:

1. *Symposium on opportunities for mathematically trained college graduates*, conducted by Professor I. S. Sokolnikoff, University of California at Los Angeles.

Participants (introduced by Professor Sokolnikoff) were Dr. William Bollay, North American Aviation, Inc.; Dr. J. B. Smyth, U. S. Navy Electronic Laboratory, San Diego; Dr. J. W. Odle, Naval Ordnance Test Station, Inyokern.

Professor Sokolnikoff set the stage by relating something of the history of the growing demand for trained mathematicians in industrial and civil-service positions. The other speakers described the opportunities in their respective fields, and the necessary training the candidate should have. They all stressed the importance of a broad background in analysis and the urgent need for training in applied mathematics in such fields as statistics, hydrodynamics, and aerodynamics, potential theory, elasticity, and electric circuit theory.

2. *The recent development of lattice theory*, by Professor Garrett Birkhoff, Harvard University.

Professor Birkhoff gave a historical description of the recent development of lattice theory. The features of a number of interesting results, many of which are due to young American mathematicians, were described. The only new result was the following relation between lattice theory and the wave question in two-dimensional space-time. A real-valued function on a lattice may be called a *valuation* if it satisfies the identity $v[x]+v[y]=v[x\cup y]+v[x\cap y]$, satisfied by dimension in projective geometry and by probability. Let space-time be partially ordered as in special relativity. Then the solutions of the wave equation are the different valuations of space-time.

3. *Averaging operators for functions*, by Professor F. H. Bohnenblust, California Institute of Technology.

Professor Bohnenblust, considered a general notion of averaging whereby the average of a function is a function (not necessarily constant). If the average of f be denoted by $\bar{f}$, it is assumed that: (1) $\overline{f_1+f_2}=\bar{f}_1+\bar{f}_2$; (2) $\overline{cf}=c\bar{f}$, if c is a constant; (3) $\bar{f}$ depends continuously (in some sense) on f; (4) $(\overline{\bar{f}g})=\bar{f}\bar{g}$; (5) $\bar{\bar{f}}=\bar{f}$. After considering an example in which $\bar{f}$ was determined by omitting certain of the harmonics in the Fourier representation of f, Professor Bohnenblust stressed the importance of condition (4) by showing how it can point the way to a standard representation of the averaging operator. Condition (5) appears then merely as an additional normalizing requirement.

4. *Collegiate mathematics in Mexico*, by Dr. Alfredo Banos, Associate Professor of Physics, University of California at Los Angeles, introduced by Professor A. E. Taylor.

An account of the development of collegiate mathematics in Mexico necessarily includes an account of the history of the National University of Mexico since the founding, in 1910, of the present day institution. Because the National University was, in its early formative years, almost exclusively a conglomeration of professional schools, such as the School of Law, the School of Medicine, the School of Engineering, and so forth, and because even today the Mexican universities do not have, properly speaking, the counterpart of the American College of Letters and Science, it turns out that an account of so-called collegiate mathematics must necessarily describe the development of the teaching of mathematics in the technical and engineering schools of the country as well as the teaching of mathematics in the Preparatory School of the National University and all other similar schools. In recent years, however, particularly after the founding of the Faculty of Sciences at the National University (1939), there has been a tendency to establish scientific curricula closely paralleling the trend in this country, and thus it is that, today, the Mexican University offers courses of instruction leading to degrees in mathematics entirely equivalent to the graduate and undergraduate degrees offered in the United States.

5. *Computing machines*, by Myron Tribus, Lecturer in Engineering, University of California at Los Angeles, introduced by Professor Clifford Bell.

The recent developments in computing machines were described and some details of the unique features of the new machines were given. It was suggested that the machines of the future may be capable of performing 20,000 to 40,000 computations per second. The machines are considered as being capable of eight basic operations: (a) programming, (b) addition, (c) subtraction, (d) multiplication, (e) division, (f) looking up in tables, (g) storage or memory, and (h) printing results. Existing machines perform these operations at slower speeds, have limited memory capacity, and require considerable time to program. The use of a binary base simplifies equipment and permits faster operation. It is expected that more attention will have to be given to the mathematics of finite difference equations and methods of solving equations, in order to utilize the full possibilities of the machines.

6. *On the interior of the convex hull of a euclidean set*, by W. S. Gustin, University of California at Los Angeles.

Let E be an arbitrary set in an n-dimensional euclidean space. It is known that any point in the convex hull of E lies in the convex hull of some subset of E containing at most $n+1$ points. It is shown that any point in the interior of the convex hull of E lies in the interior of the convex hull of some subset of E containing at most $2n$ points.

P. H. DAUS, *Secretary*

APRIL MEETING OF THE OHIO SECTION

The thirty-first annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 3, 1947. Professor S. A. Rowland, Chairman of the Section, presided.

Eighty persons registered attendance, including the following fifty-four members of the Association: W. E. Anderson, F. R. Bamforth, Grace M. Bareis, H. M. Beatty, Foster Brooks, V. B. Caris, F. E. Carr, A. B. Carson, E. H. Clarke, Florentina M. Clinton, Rufus Crane, Wayne Dancer, R. H. Downing, H. P. Fawcett, H. E. Fettis, Frances Freese, B. E. Gatewood, B. C. Glover, E. L. Godfrey, L. J. Green, C. H. Heinke, R. G. Helsel, R. C. Hildner, L. A. Jehn, Margaret E. Jones, L. C. Knight, A. C. Ladner, S. W. McCuskey, Margaret Mauch, H. E. Menke, E. J. Mickle, C. V. Newsom, H. C. Parrish, H. S. Pollard, J. F. Randolph, S. E. Rasor, Hortense Rickard, R. F. Rinehart, L. D. Rodabaugh, Louis Ross, S. A. Rowland, K. C. Schraut, Samuel Selby, G. W. Starcher, H. E. Stelson, Irving Sussman, H. S. Toney, E. P. Vance, R. W. Wagner, D. R. Whitney, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, C. R. Wylie, Jr., P. M. Young.

The following officers were elected for the coming year: Chairman, H. S. Pollard, Miami University; Secretary-Treasurer, Foster Brooks, Kent State University; Member of Executive Committee, E. J. Mickle, Ohio State University; Member of Program Committee, R. B. Wildermuth, Capital University. It is expected that the next meeting will be held on Saturday, April 3, 1948.

The following program was presented:

1. *The Association's interest in pre-college training*, by Professor S. A. Rowland, Ohio Wesleyan University.

This paper is concerned especially with the situation in Ohio, but is doubtless applicable elsewhere. For a good many years the standard of achievement of pre-college students in mathematics has been deteriorating. The state-supported colleges and universities do not have complete freedom of action to maintain high standards of admission since by law they must admit graduates of approved high schools, and the approval of high schools is not controlled by them. Few private colleges have enough endowment to be independent as to standards of admission.

This Association has fostered high achievement in college mathematics, but faulty preparation for college is now undermining some of our best efforts. Possibly our most important task now is to embark upon an educational campaign, and to establish cooperation with the Ohio College Association and with other educational organizations in an endeavor to get the legislature to establish a State College Entrance Board independent of the State Department of Education. Such a board would not attempt to dictate the requirements for a high school diploma, but would set up entrance examinations which would ensure reasonable uniformity of preparation for college. The high schools could provide an intensive fifth year for such students as needed it and wanted it in order to pass the college entrance examinations.

2. *A proof of five-color sufficiency*, by Professor L. D. Rodabaugh, University of Akron.

Use is made in this paper of the well-known duality between regions and

common segments of their boundaries on the one hand, and points and their joins (no two points being permitted to have more than one join) on the other. There is described a *completion procedure*, which is to be applied to any admissible network of points and joins, the result being defined as a *complete network*. There follow the statements and proofs of nine properties possessed by every complete network, after which it is easy to prove the five-color sufficiency.

The proof does not depend on the formula of Euler which states that $V-E+F=2$, and has immediate application to any set of regions simply and completely covering the plane or sphere, regardless of the order of connectivity of any of the regions. The techniques used in this paper can, however, be used to prove this formula for any set of simply connected regions simply and completely covering the sphere. If the four-color problem be considered, the techniques employed can be used to prove many of the properties of the so-called "first non-colorable map." One illustration is given.

3. *An experimental evaluation of the base e*, by Professor H. S. Pollard, Miami University.

Buffon's needle problem deals with the probability that a needle, thrown at random on a table ruled with parallel lines, will cross one of these lines. This probability is a function of the length of the needle, the distance between the rulings, and π. By determining the relative frequency of crossings, many experimental calculations of the value of π have been made.

This paper discusses a method of evaluating experimentally the number e. When a needle, which rotates about its mid-point, comes to rest, its direction will locate a definite point x on an axis which does not pass through the mid-point of the needle. The probability density of the variable x is given by the expression $1/[\pi(1+x^2)]$. The mathematical expectation of $\phi(x)=\cos x$ is then

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx$$

whose value may be shown to be e^{-1}. Hence e^{-1} may be evaluated by making successive spins of the needle and calculating the mean of the values of $\cos x$ at the points x so located.

4. *A proof of Wilson's theorem*, by Robert E. Seall, Denison University, introduced by Professor F. B. Wiley.

The theorem $(p-1)^k+1\equiv 0 \pmod p$, p any prime, is established as a corollary to two lemmas.

LEMMA 1. *If n is any positive integer prime to each of the integers 2 to k*, then

$$\sum_{i=1}^{n-1} i^{k-1} \equiv 0 \pmod n.$$

The proof of this comes from writing $2^k, 3^k, \cdots, n^k$ as $(1+1)^k, (1+2)^k, \cdots, (1+n-1)^k$, expanding each by the binomial theorem, and then summing all to

obtain

$$\frac{k}{1}\sum i + \frac{k(k-1)}{2!}\sum i^2 + \cdots + \frac{k}{1}\sum i^{k-1} = n^k - n.$$

Now give k the successive values 2 to k in this last expression, and then solve the resulting system of equations, linear in the $\sum$'s for $\sum i^{k-1}$.

LEMMA 2. *If $S_{n,k}$ represents the sum of all possible products of k unlike factors chosen from the positive integers 1 to n, then*

$$kS_{n,k} = S_{n,k-1}\sum i - S_{n,k-2}\sum i^2 + \cdots + (-1)^{k+1}\sum i^k.$$

The truth of this may be established by beginning with the last term and adding each preceding term in turn.

5. *Intuition in mathematics*, by Professor Tibor Radó, Ohio State University

In teaching, as well as in research, the initial reaction of a person toward a specific mathematical issue is largely determined by what may be termed intuition. From this point of view, intuition is an important component of motivation for the student, the teacher, and the research man. These general remarks were illustrated with a number of specific examples.

6. *The teacher training program for teachers in elementary schools*, by Professor H. P. Fawcett, Ohio State University.

This was a report from a committee that has been working on this problem for some time. It is hoped that some changes may be effected in the program which will strengthen the preparation of the teacher in mathematics.

RUFUS CRANE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Muncie, Oct. 17, 1947
IOWA, Fairfield, April 1, 1948
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Bryn Mawr, Pa., November 29, 1947
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
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TEXAS
UPPER NEW YORK STATE
WISCONSIN, Beloit, May 8, 1948

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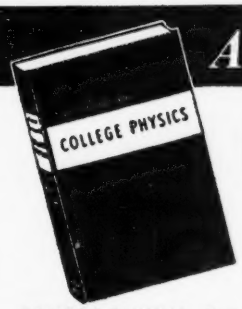

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CONTENTS

What is Cantor's Continuum Problem? . . . . . Kurt Gödel 515
Personnel and Training Problems in Statistics . . . S. S. Wilks 525
A Simple Analytic Proof of a General χ^2 Theorem . C. L. Weaver 529
A Family of Integers and a Theorem on Circles. . . . . G. T. Williams and D. H. Browne 534
Mathematical Notes . . Victor Thébault, Lucien Droussent 537
Classroom Notes. . . . . L. V. Robinson, L. C. Green, K. W. Folley, L. S. Laws, J. F. Heyda 540
Elementary Problems and Solutions 545
Advanced Problems and Solutions. 549
Recent Publications. 553
Clubs and Allied Activities. 558
News and Notices 561
The Mathematical Association of America 563
April Meeting of the Rocky Mountain Section 563
April Meeting of the Louisiana-Mississippi Section 566
April Meeting of the Southwestern Section 568
April Meeting of the Iowa Section 570
Calendar of Future Meetings 572

NOVEMBER 1947